Musterlösung Serie 7

Set Models

22. (b) Consider $\mathbf{M} = (\omega_{\mathbf{1}}, \in)$. Following the enumeration in Chapter 3:

ZFC_0 : We do have the *real* empty set $\varnothing^{\mathbf{V}} \in \omega_1$.

ZFC_1 : ω_1 is transitive, and every transitive subset of $\mathbf{V}$ is extensional. Alternatively, one can argue that for every x, y distinct elements of ω_1 we have either $x \in y$ or $y \in x$, which again proves extensionality.

ZFC_2 : The Axiom of Pairing does not hold, as for instance there is no element of ω_1 which $\mathbf{M}$ *sees* as $\{0, 2\}$. More formally,

$$\mathbf{M} \models \nexists x \, \forall y \, (y \in x \leftrightarrow (y = 0 \vee y = 2)).$$

ZFC_3 : We have that $\bigcup \varnothing = \varnothing$ and that for all elements $\alpha \in \omega_1$, $\bigcup(\alpha + 1) = \alpha$.

ZFC_4 : Clearly $\omega \in \omega_1$.

ZFC_5 : The Axiom Schema of Separation does not hold. Similarly to ZFC_2 :, we have that

$$\mathbf{M} \models \nexists x \, \forall y \, (y \in x \leftrightarrow (y \in 3 \wedge (y = 0 \vee y = 2))).$$

ZFC_6 : From the fact that

$$\mathbf{M} \models \forall x \, \forall y \, (x \subseteq y \leftrightarrow x \in y),$$

it follows that $\mathbf{M} \models \forall x \, (\mathcal{P}(x) = x \cup \{x\})$.

ZFC_7 : Since ω_1 is regular, given an element $\mu \in \omega_1$ and a definable function $f \colon \mu \to \omega_1$, we can find an upper bound $\lambda \in \omega_1$ such that for all $\alpha \in \mu$ we have that $f(\alpha) < \mu$. This proves the Axiom Schema of Replacement in the form in which it appears in Chapter 3 of the book.

ZFC_8 : Every subset of $\mathbf{V}$ (with the standard $\in$ relation) satisfies the Axiom of Foundation, hence $\mathbf{M}$ does as well.

ZFC_9 : For any $\mathcal{F}$, we have that $C = \{\varnothing\}$ satisfies the Axiom of Choice in the form appearing in the statement, as for all $x \in \omega_1 \setminus \{\varnothing\}$ we have that $\varnothing \in x$.