Musterlösung Serie 8

ON THE EQUALITY $\kappa^\omega = \kappa$

We will repeadetly use the fact that for every infinite cardinal λ, $\lambda^\lambda = 2^\lambda$, which can be proved for instance using the fact that $\lambda \cdot \lambda = \lambda$.

23. Since it is an idea that can be applied in other cases as well, we leverage the fact that ω_1 is regular, and hence every sequence of length ω admits a countable upper bound in ω_1. For each sequence $s\colon \omega \to \omega_1$, let μ_s be the least countable infinite ordinal such that for all $n \in \omega$, $s(n) < \mu_s$. Consider now the partition of ${}^\omega\omega_1$ given by the classes, for $\alpha \in \omega_1 \setminus \omega$,

$$D_\alpha = \{s \in {}^\omega\omega_1 : \mu_s = \alpha\}.$$

There are ω_1 such classes and each class has cardinality $\mathfrak{c}$, since every $\alpha \in \omega_1 \setminus \omega$ is countable and $\omega^\omega = 2^\omega = \mathfrak{c}$. It follows that $\omega_1^\omega = \omega_1 \cdot \mathfrak{c} = \mathfrak{c}$, where the last equality is guaranteed by the fact that $\mathfrak{c}$ is uncountable.

25. Given the argument shown in the previous exercise, we quickly prove the claim by induction on $n \in \omega$, where the base step is the fact that $\omega_0^\omega = \omega^\omega = 2^\omega = \mathfrak{c} = \omega_0 \cdot \mathfrak{c}$. Assume now that the claim holds for some $n \in \omega$. Then, since ω_{n+1} is regular and in particular $\mathrm{cf}(\omega_{n+1}) > \omega$, following the same reasoning as in Ex. 23 we can deduce that $\omega_{n+1}^\omega = \omega_{n+1} \cdot \omega_n^\omega$. We now use the induction hypothesis and conclude $\omega_{n+1} \cdot \omega_n^\omega = \omega_{n+1} \cdot \omega_n \cdot \mathfrak{c} = \max(\omega_{n+1}, \omega_n, \mathfrak{c}) = \max(\omega_{n+1}, \mathfrak{c})$, which is what we wanted to show.

24 (c) We give an argument that can be generalized to give an alternative proof of the fact that for any infinite cardinal λ, we have $\lambda^{\mathrm{cf}(\lambda)} > \lambda$. Let ω_α be a cardinal and consider $\kappa = \omega_{\alpha+\omega}$, so that we clearly have $\mathrm{cf}(\kappa) = \omega$. Let $f\colon \kappa \to \kappa^\omega$ be a function. We will use a diagonalization argument to show that f can not be surjective. Indeed, let $(\alpha_n)_{n\in\omega}$ be a countable sequence of ordinals which is cofinal in κ. For each $n \in \omega$, since $|\alpha_n| < \kappa$, we can find an ordinal $\beta_n \in \kappa$ such that for all $\alpha \in \alpha_n$, we have $(f(\alpha))(n) \neq \beta_n$. Since $(\alpha_n)_{n\in\omega}$ is cofinal, by construction there can not be any element $\beta \in \kappa$ such that $f(\beta) = (\beta_n)_{n\in\omega}$, which implies that f can not be surjective, and hence $\kappa < \kappa^\omega$.