Number Theory II : Intro. to Modular Forms

$\boxed{\text{Exercise Class \# 1}}$ (3 / March / 2023)

In the first part of this class we will review some concepts from Lectures 1 to 3, in relation to the Exercise Sheet 1

① Classical modular forms are functions $f : \mathbb{H} \to \mathbb{C}$ that transform nicely w.r.t. the action of $SL_2(\mathbb{Z})$ (or $\Gamma \leq SL_2(\mathbb{Z})$, finite index subgroup) on $\mathbb{H}$. Hence, we need to understand this action.

This is a restriction of the left action

$$GL_2(\mathbb{C}) \times \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}} \qquad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

$$\Big(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{= \gamma}, z\Big) \longmapsto \gamma \circ z = \underbrace{\frac{az+b}{cz+d}}_{\text{"Linear Fractional Transformation" (TFL)}}$$

with the usual conventions regarding $\infty \in \hat{\mathbb{C}}$

"Action" means : i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ z = z \quad \forall z \in \hat{\mathbb{C}}$

ii) $\gamma_1 \circ (\gamma_2 \circ z) = (\gamma_1 \gamma_2) \circ z \quad \forall \gamma_1, \gamma_2 \in GL_2(\mathbb{C})$, $\forall z \in \hat{\mathbb{C}}$

This action can be restricted to $SL_2(\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H}$ (Lecture 2)

Exercise 1 is about the action $GL_2(\mathbb{R}) \times (\mathbb{C} \setminus \mathbb{R}) \longrightarrow (\mathbb{C} \setminus \mathbb{R})$

Note that $GL_2(\mathbb{R})$ does not act on $\mathbb{H}$ since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ z = -z \quad \text{hence} \quad \mathbb{H} \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}]{} \mathbb{H}^- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$$

② Regarding Exercise 2 : recall from Lecture 1

$$\mathcal{L} := \{L \leq \mathbb{C} \text{ lattice}\} = \{\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} : \{\omega_1, \omega_2\} \subseteq \mathbb{C} \text{ l.i. over } \mathbb{R}\}$$

General definition : A lattice in a locally compact group G (Example $G = (\mathbb{C}, +)$) is a discrete subgroup H s.t. G/H has finite volume w.r.t. some G-invariant Borel measure

Recall $\mathbb{C}^\times \circlearrowright \mathcal{L}$ by multiplication

The map $\mathbb{H} \longrightarrow \mathcal{L}$ induces $SL_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{C}^\times\backslash\mathcal{L}$
$\tau \longmapsto \tau\mathbb{Z}+\mathbb{Z}$ bijection

Exercise 2 : For $k\in\mathbb{Z}$ prove that there is bijection

$$\left\{ f:\mathbb{H}\to\mathbb{C} \text{ s.t. } f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau) \;\; \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}), \forall \tau\in\mathbb{H} \right\} \longrightarrow \left\{ \begin{array}{l} F:\mathcal{L}\to\mathcal{L} \text{ s.t.} \\ F(\lambda L)=\lambda^{-k}F(L) \\ \forall L\in\mathcal{L}, \forall\lambda\in\mathbb{C}^\times \end{array}\right\}$$

$F \longmapsto f_F$, $f_F(\tau) = F(\tau\mathbb{Z}+\mathbb{Z})$

$f \longmapsto F_f$, $F_f(L) = F_f(\omega_1\mathbb{Z}+\omega_2\mathbb{Z}) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right)$

if $L=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, $\frac{\omega_1}{\omega_2}\in\mathbb{H}$

Must show : f_F transforms like weight k modular form for $SL_2(\mathbb{Z})$

F_f is well defined

F_f transforms like "degree $-k$"

$f\longmapsto F_f$ and $F\longmapsto f_F$ are inverse maps to each other

3 We recall classification of LFT $\gamma\in SL_2(\mathbb{R})$, $\gamma\neq\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ (Lecture 2)

according to $tr(\gamma)$ or its fixed points in $\hat{\mathbb{C}}$

$$\gamma\circ z = \frac{az+b}{cz+d} = z$$

If $z\neq\infty$ this is ~~Equi~~ valent to $cz^2+(d-a)z-b=0$ ☆

quadratic if $c\neq0$ of

discriminant $\Delta = (d-a)^2+4cb$

$$= (d+a)^2-4da+4cb$$

$$= tr(\gamma)^2-4\underbrace{(ad-bc)}_{1}$$

$$= tr(\gamma)^2-4$$

Remark : $\gamma\circ\infty = \frac{a}{c}$ hence $\gamma\circ\infty=\infty \iff c=0$.

$\boxed{\text{Parabolic}}$ $|tr(\gamma)| = 2 \iff \Delta = 0$

$\Rightarrow$ unique fixed point in $\hat{\mathbb{C}}$

Indeed: If $c = 0$ then ∞ is fixed

Also (*) becomes $(d-a)z - b = 0$. But $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$a + d = \pm 2$, $ad = 1 \Rightarrow a + \frac{1}{a} = \pm 2$

$\iff a^2 \mp 2a + 1 = 0$

$\iff (a \mp 1)^2 = 0$

$\iff$ $a = 1$, $d = 1$, $b \neq 0$ or $a = -1$, $d = -1$, $b \neq 0$

So no fixed point in $\mathbb{C}$ and $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, $b \neq 0$

If $c \neq 0$ then ∞ is not fixed and (*) has <u>one</u> solution

$x_0 = \dfrac{a-d}{2c} \in \mathbb{R}$ $\rightsquigarrow$ unique fixed point in $\hat{\mathbb{C}}$

In this case $M := \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{R})$ sends $\infty \longmapsto x_0$

hence $M^{-1}\gamma M$ is parabolic w/ fixed point ∞

$\Rightarrow M^{-1}\gamma M = \begin{pmatrix} \pm 1 & b' \\ 0 & \pm 1 \end{pmatrix}$ for some $b' \in \mathbb{R}$

$\boxed{\text{Elliptic}}$ $|tr(\gamma)| \in [0, 2[\iff \Delta < 0$

$\Rightarrow$ two fixed points $\tau, \bar{\tau}$ w/ $\tau \in \mathbb{H}$

$$\tau = \frac{a - d + i\sqrt{4 - tr(\gamma)^2}}{2c}$$

Note that $c \neq 0$ (∞ not fixed) since $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ has

$\left| \text{trace}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right| = \left| a + \frac{1}{a} \right| \geq 2$ so it is <u>never</u> elliptic!

$\left(\begin{array}{l} \iff a^2 + 1 \geq 2|a| \\ \iff (|a| - 1)^2 \geq 0 \end{array} \right)$

If $\tau = x + iy$, $y > 0$ then $M := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$

$\in SL_2(\mathbb{Z})$

sends $i \longmapsto x + iy = \tau$ and

$M^{-1} \gamma M$ is elliptic w/ fixed points $i, -i$

$$\begin{matrix} \| \\ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{R}) \end{matrix} \qquad \frac{a' i + b'}{c' i + d'} = i \iff b' = -c', \ a' = d'$$

$$\Rightarrow M^{-1} \gamma M = \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix}, \quad (a')^2 + (b')^2 = 1 \text{ so } a' = \cos(\theta), \ b' = \sin(\theta) \text{ where } \theta \in \mathbb{R}$$

$$\Rightarrow M^{-1} \gamma M = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \longrightarrow \text{rotation matrix}$$

$\boxed{\text{Hyperbolic}}$: $|tr(\gamma)| \in]2, \infty[\iff \Delta > 0$

$\Rightarrow$ two fixed points in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

Indeed: if $c = 0$ then ∞ is fixed

and $\gamma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ $\quad a \in \mathbb{R}$, $a \neq \pm 1$

acts as $z \longmapsto a^2 z + ba$ with $x_0 := \frac{ba}{1 - a^2} \in \mathbb{R}$ fixed

In this case $M := \begin{pmatrix} 1 & +x_0 \\ 0 & 1 \end{pmatrix}$ sends $0 \longmapsto x_0$ and $\infty \longmapsto \infty$

$M^{-1} \gamma M = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ is hyperbolic w/ fixed points $0, \infty$

so $b' = 0$, $c' = 0$

$$\Rightarrow M^{-1} \gamma M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R}, \ \lambda \neq 1, -1, 0.$$

If $c \neq 0$ then ∞ is not fixed and γ has fixed points

$$x_1 = \frac{a - d + \sqrt{tr(\gamma) - 4}}{2c}, \quad x_2 = \frac{a - d - \sqrt{tr(\gamma) - 4}}{2c}$$

both real and $x_1 - x_2$.

Put $M := \frac{1}{\sqrt{x_1 - x_2}} \begin{pmatrix} -x_2 & x_1 \\ -1 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$ sending $0 \mapsto x_1$, $\infty \mapsto x_2$

hence $M^{-1}\gamma M$ hyperbolic w/ fixed points $0, \infty$

$\Rightarrow M^{-1}\gamma M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{R}, \lambda \neq 0, 1, -1$

In Exercise 3 you have to analyse invariant subsets $\subseteq \mathbb{R}^2$ of hyperbolic / elliptic / parabolic transformations. To make life easier note that: for $Y \subseteq \mathbb{R}^2$, $M \in SL_2(\mathbb{R})$

Y invariant under $\gamma \Leftrightarrow M^{-1}(Y)$ invariant under $M^{-1}\gamma M$

(easier to analyse for smart choice of M)

(4) Recall $SL_2(\mathbb{Z}) = \langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ (Lecture 3) and there is an algorithm to write any $\gamma \in SL_2(\mathbb{Z})$ in terms of S and T (see lecture notes!)

(5) Useful general fact: If $\Gamma \leq SL_2(\mathbb{Z})$ finite index subgroup

$\tilde{\Gamma} := \Gamma \cup \Gamma \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (this is just Γ if $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, otherwise $\underbrace{\Gamma \leq \tilde{\Gamma}}_{\text{index } 2} \leq SL_2(\mathbb{Z})$)

and $SL_2(\mathbb{Z}) = \bigsqcup_{i=1}^{m} \tilde{\Gamma}\gamma_i$, $m = [SL_2(\mathbb{Z}) : \tilde{\Gamma}]$

then for any fund. domain F of $SL_2(\mathbb{Z})$ the set $F' := \bigcup_{i=1}^{m} \gamma_i(F)$ is a fund. domain for Γ

Recall: $F = \overline{F^\circ}$, $\mathbb{H} = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma(F)$ and $\gamma(F^\circ) \cap F^\circ \neq \emptyset \Rightarrow \gamma = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

To prove that F' is a fundamental domain for Γ we have to check

i) $\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma(F')$

ii) $\gamma \in \Gamma$, $\gamma(F')^\circ \cap (F')^\circ \neq \emptyset \Rightarrow \gamma = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i) is easy: $\mathbb{H} = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma(F) = \bigcup_{\gamma \in \tilde{\Gamma}} \bigcup_{i=1}^{m} (\gamma\gamma_i)(F)$

$= \bigcup_{\gamma \in \tilde{\Gamma}} \gamma(F') = \bigcup_{\gamma \in \Gamma} \gamma(F')$

ii) assume $\gamma(F')^\circ \cap (F')^\circ \neq \emptyset$ with $\gamma \in \Gamma$.

Note that $\bigcup_{i=1}^{m} \gamma_i(F^\circ)$ open and dense in $(F')^\circ$

hence $\bigcup_{i=1}^{m} \gamma\gamma_i(F^\circ)$ " " " " $\gamma(F')^\circ$

$\Rightarrow \left(\bigcup_{i=1}^{m} \gamma_i(F^\circ)\right) \cap \left(\bigcup_{i=1}^{m} \gamma\gamma_i(F^\circ)\right) \neq \emptyset$

i.e $\exists\ i,j \in \{1, \dots, m\}$ s.t. $\gamma_i(F^\circ) \cap \gamma\gamma_j(F^\circ) \neq \emptyset$

$\Rightarrow \gamma_i^{-1}\gamma\gamma_j(F^\circ) \cap F^\circ \neq \emptyset$

$\Rightarrow \gamma_i^{-1}\gamma\gamma_j = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \gamma\gamma_j = \pm\gamma_i$

$\Rightarrow i = j$ and $\gamma = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ //