Exercise class #2 (17/March/2023)

We will review the exercises of the second sheet.

(1) We want to prove

(1) $$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}' \left(\frac{1}{z+n} - \frac{1}{z}\right), \quad z\in\mathbb{C}\setminus\mathbb{Z}$$

and deduce (2) $\zeta(k) = -\frac{(2\pi i)^k}{2(k!)} B_k$ for $k\in\mathbb{Z}^+$ even

Here $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s\in\mathbb{C}$ w/ $\mathrm{Re}(s)>1$ (Riemann's zeta)

and (3) $\frac{z}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_k \cdot z^k}{k!}$, B_k is called the k-th Bernoulli number

These formulas were used in Lecture 5 when computing the Fourier expansion (at the cusp $i\infty$) of the Eisenstein Series G_k, $k>2$ even.

(1.a) We start by proving:

(4) $$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right), \quad z\in\mathbb{C}\setminus\{0\}$$

Recall: $$\sin(z) := \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos(z) := \frac{e^{iz}+e^{-iz}}{2}$$

$\tan(z) := \frac{\sin(z)}{\cos(z)}$, $\cot(z) := \frac{\cos(z)}{\sin(z)}$, other trigonometric functions..

Recall: (Infinite products) see "Complex Analysis" S. Lang (Springer 1999) Chapter XIII

$\boxed{\text{Lemma}}$: Let $(f_n)_{n\geq 1}$ be a sequence of analytic functions on an open set $U \subseteq \mathbb{C}$. Write $f_n(z) = 1 + h_n(z)$ and assume that the series $\sum_n h_n(z)$ converges uniformly and absolutely on U. Let K be a compact subset of U not containing any of the zeroes of the functions f_n for all n. Then the product $\prod_n f_n$ converges to an analytic function f on U, for $z \in K$ we have

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)}$$

("logarithmic derivative")

and the convergence is absolute and uniform on K.

For $\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)$ we have $\sum_{n=1}^{\infty}\left|\frac{z^2}{\pi^2 n^2}\right| = \frac{|z|^2}{\pi^2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\text{converges}}$

hence $\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)$ is entire (holo. in $\mathbb{C}$)

and one can take logarithmic derivative.

To prove (4)

$$\frac{\sin(z)}{z} = \frac{e^{iz} - e^{-iz}}{2iz} = \lim_{n\to\infty} f_n(z)$$

$$f_n(z) = \frac{\left(1 + \frac{iz}{n}\right)^n - \left(1 - \frac{iz}{n}\right)^n}{2iz}$$

since $e^{w} = \lim_{n\to\infty}\left(1 + \frac{w}{n}\right)^n, \ w \in \mathbb{C}$

Next, one proves $f_n(z) = \prod_{k=1}^{m} \left(1 - \frac{z^2}{n^2}\left(\frac{1+\cos\left(\frac{2k\pi}{n}\right)}{1-\cos\left(\frac{2k\pi}{n}\right)}\right)\right)$ $(\star)$

if $n = 2m+1$, $m \in \mathbb{Z}^+$ (so n odd)

To see this note that $f_n(z)$, for n odd, is a polynomial in z with constant term 1 and degree $n-1$:

$$f_n(z) = \frac{\left(\cancel{1} + iz + \dots + \frac{i^n z^n}{n^n}\right) - \left(\cancel{1} - iz + \dots - \frac{i^n z^n}{n^n}\right)}{2iz}$$

$$= 1 + \dots + \frac{i^{n-1} z^{n-1}}{n^n}$$

The RHS of $(\star)$ is also a poly in z with constant term 1 and degree $2m = n-1$ (note that $\cos\left(\frac{2k\pi}{n}\right) \in]0,1[$ for $k \in \{1,\dots,m\}$, $n=2m+1$)

To prove $(\star)$ compute the roots of $f_n(z)$

$$f_n(z) = 0 \iff \left(1 + \frac{iz}{n}\right)^n = \left(1 - \frac{iz}{n}\right)^n \quad (\text{since } z \neq 0)$$

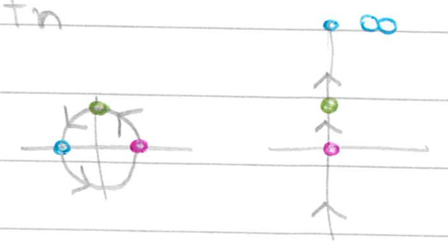
$$\iff \frac{\left(1 + \frac{iz}{n}\right)}{\left(1 - \frac{iz}{n}\right)} = \xi \quad n\text{-th root of unity}, \ \xi \neq 1$$

$$\iff (1+\xi)\frac{iz}{n} = \xi - 1$$

$$\iff z = -in\left(\frac{\xi - 1}{\xi + 1}\right)$$

This gives the $n-1$ roots of f_n

Rmk: $z \longmapsto \frac{z-1}{z+1}$ maps



$$1 \longmapsto 0$$

$$i \longmapsto \frac{i-1}{i+1} = \frac{(i-1)(1-i)}{1+1} = \frac{i + \cancel{1} - \cancel{1} + i}{2} = i$$

$$-1 \longmapsto \infty$$

$$\Longrightarrow \quad f_n(z) = \prod_{\substack{\zeta^n = 1 \\ \zeta \neq 1}} \left(1 - \frac{iz}{n}\left(\frac{\zeta+1}{\zeta-1}\right)\right)$$

$$\zeta = \exp(2\pi i k/n) \qquad k = 1, \dots, n-1 = 2m$$

$$f_n(z) = \prod_{k=1}^{m} \left(1 - \frac{iz}{n}\left(\frac{\exp(2\pi i k/n)+1}{\exp(2\pi i k/n)-1}\right)\right)\left(1 - \frac{iz}{n}\left(\frac{\exp(-2\pi i k/n)+1}{\exp(-2\pi i k/n)-1}\right.\right.$$

This implies (☆)

We get

$$\frac{\sin(z)}{z} = \lim_{\substack{n \to \infty \\ n \text{ odd}}} \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{z^2}{n^2}\left(\frac{1+\cos(2k\pi/n)}{1-\cos(2k\pi/n)}\right)\right)$$

But

$$\lim_{n \to \infty} \frac{1}{n^2}\left(\frac{1+\cos(2k\pi/n)}{1-\cos(2k\pi/n)}\right) = \frac{1}{\pi^2 k^2} \qquad \text{(check this)}$$

hence

$$\frac{\sin(z)}{z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 k^2}\right) \qquad \text{(check convergence)}$$

(1.b) Taking logarithmic derivatives

$$\underbrace{\frac{\cos(z)\, z - \sin(z)}{z^2} \cdot \frac{z}{\sin(z)}} = \sum_{n=1}^{\infty}\left(\frac{-2z}{n^2\pi^2}\right)\left(1 - \frac{z^2}{n^2\pi^2}\right)^{-1}$$

$$\cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{-2z}{(n^2\pi^2 - z^2)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n\pi+z} - \frac{1}{\pi n}\right) - \left(\frac{1}{n\pi - z} - \frac{1}{\pi n}\right)$$

This implies (1)

1.c and 1.d See Lecture 5 notes. One gets

$$\pi z \cot(\pi z) = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k} \quad \left(\text{using } (1)\right)$$

for $|z|<1$,

and

$$\pi z \cot(z) = B_0 + \pi i (1+2B_1) z + \sum_{k=2}^{\infty} \frac{(2\pi i)^k}{k!} B_k \; z^k$$

also for $|z|<1$.

This implies $B_0 = 1$, $B_2 = -1/2$, $B_k = 0$ for $k>1$ odd

and $-2\zeta(k) = \frac{(2\pi i)^k}{k!} B_k$ for $k>1$ even.

2 Recall (Lecture 5)

$$G_k(\tau) = -2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad , \quad k>2 \text{ even}$$

where $\sigma_s(n) = \sum_{\substack{d|n \\ d>0}} d^s$, $q := \exp(2\pi i \tau)$

For $k=2$ we define

$$G_2(\tau) = -2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$= -\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n \quad \left(\text{using } \zeta(2) = \frac{\pi^2}{6}\right)$$

In contrast with the case $k>2$, the function G_2 is not modular

It satisfies $G_2(\tau+1) = G_2(\tau)$ (because of the Fourier expansion)

but $G_2(-1/\tau) = \tau^2 G_2(\tau) \underbrace{- 2\pi i \tau}_{\text{extra term that kills modularity}}$

(modular wt. 2 would give $G_2(-1/\tau) = \tau^2 G_2(\tau)$)

One can prove $G_2(\tau) = F_1(\tau) := \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}' \frac{1}{(m+n\tau)^2}$ where it is assumed that $(m,n)\neq(0,0)$

but the double series is not abs. convergent hence it cannot be re-arranged at will.

Define $F_2(\tau) := \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}}' \frac{1}{(m+n\tau)^2}$ $\left(\neq F_1(\tau)\right)$

$$\left.\begin{aligned} H_1(\tau) &:= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}' \frac{1}{(m-1+n\tau)(m+n\tau)} \\ H_2(\tau) &:= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}}' \frac{1}{(m-1+n\tau)(m+n\tau)} \end{aligned}\right\} (m,n)\neq(0,0),(1,0)$$

2.a Want to prove (5) $H_1(\tau) = 2$, (6) $H_2(\tau) = 2 - \frac{2\pi i}{\tau}$

(7) $F_1(\tau) - H_1(\tau) = F_2(\tau) - H_2(\tau)$

To prove (5) write

$$\frac{1}{(m-1+n\tau)(m+n\tau)} = \frac{1}{(m-1+n\tau)} - \frac{1}{(m+n\tau)}$$

If $n\neq 0$ we have $\sum_{m\in\mathbb{Z}} \frac{1}{(m-1+n\tau)} - \frac{1}{(m+n\tau)} = 0$ (telescoping series)

If $n=0$ we have $\sum_{\substack{m\in\mathbb{Z} \\ m\neq 0,1}}' \frac{1}{(m-1)} - \frac{1}{m} = \underbrace{\sum_{m\leq -1}'}_{\text{telescoping series}} + \underbrace{\sum_{m\geq 2}'}_{\text{telescoping series}} = 1+1=2$

$$\Rightarrow H_1(\tau) = 2$$

To compute $H_2(\tau)$ we first assume $m \neq 0,1$ and write

$$\sum_{n\in\mathbb{Z}} \frac{1}{m-1+n\tau} - \frac{1}{m+n\tau} = \tau^{-1} \sum_{n\in\mathbb{Z}} \frac{1}{\tau^{-1}(m-1)+n} - \frac{1}{\tau^{-1}m+n}$$

$$= \tau^{-1}\left(\sum_{n\neq 0}\left(\frac{1}{\tau^{-1}(m-1)+n} - \frac{1}{n}\right) - \sum_{n\neq 0}\left(\frac{1}{\tau^{-1}m+n} - \frac{1}{n}\right)\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right)$$

$$= \tau^{-1}\left(\pi\cot\left(\pi\frac{(m-1)}{\tau}\right) - \frac{1}{\tau^{-1}(m-1)} - \pi\cot\left(\pi\frac{m}{\tau}\right) + \frac{1}{\tau^{-1}m}\right) + \frac{1}{m-1} -$$

$$= \tau^{-1}\left(\pi\cot\left(\pi\frac{(m-1)}{\tau}\right) - \pi\cot\left(\frac{\pi m}{\tau}\right)\right)$$

Similarly

$$\sum_{m=0,1} \sum_{n\neq 0}' \frac{1}{m-1+n\tau} - \frac{1}{m+n\tau} = \tau^{-1}\left(\pi\cot\left(\frac{\pi(-1)}{\tau}\right) - \pi\cot\left(\frac{\pi\cdot 1}{\tau}\right)\right) + 2$$

(check this!)

Hence $$H_2(\tau) = \tau^{-1}\left(\lim_{N\to-\infty} \pi\cot\left(\frac{\pi(N-1)}{\tau}\right) - \lim_{M\to\infty}\pi\cot\left(\frac{\pi M}{\tau}\right)\right) + 2$$

To compute the limits use

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = i\left(\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right) = i\left(\frac{e^{2iz}+1}{e^{2iz}-1}\right),$$

and $|e^{2iz}| = e^{-2\operatorname{Im}(z)}$, so: $\operatorname{Im}(z)\to\infty \Rightarrow \cot(z)\to -i$

$\operatorname{Im}(z)\to-\infty \Rightarrow \cot(z)\to i$

Hence $$H_2(\tau) = \tau^{-1}\pi(-i - i) + 2 = 2 - \frac{2\pi i}{\tau}$$

Finally, to prove (7) note that

$$\frac{1}{(m+n\tau)^2} - \frac{1}{(m-1+n\tau)(m+n\tau)} = \underbrace{\frac{-1}{(m+n\tau)^2(m-1+n\tau)}}$$

it is absolutely summable over (n,m) hence the corresponding series can be re-arranged at will

(2.b) Easy to show that (1.a) $\Rightarrow$ $F_1 - F_2 = H_1 - H_2 = \frac{2\pi i}{\tau}$ (8)

Now, to prove $F_1\left(-\frac{1}{\tau}\right) = \tau^2 F_1(\tau) - 2\pi i\tau$ one first computes

$$F_1\left(-\frac{1}{\tau}\right) = \sum_n \sum_m{}' \frac{1}{\left(m-\frac{n}{\tau}\right)^2} = \tau^2 \underbrace{\sum_n \sum_m \frac{1}{(\tau m - n)^2}}_{F_2(\tau)}$$

But F_2 can be written in terms of F_1 by (8)

(2.c) The equality $F_1(\tau) = G_2(\tau)$ follows from computing the Fourier expansion of $F_1(\tau)$ as in Lecture 5

(3) To show identities between modular forms, one uses dimension formulas for M_k (See Lecture 6)

Recall: $E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \frac{1}{2\zeta(k)} G_k(\tau) \in M_k$

normalized Eisenstein series $(k>2$ even$)$

↳ constant Fourier coeff. $= 1$

Example of identity: $E_4^2 = E_8$ because $E_4^2, E_8 \in M_8 \setminus \{0\}$ and $\dim_{\mathbb{C}} M_8 = 1$ (see Lecture 6) hence $E_4^2 = \lambda E_8$, $\lambda \in \mathbb{C}^\times$

$$\Rightarrow \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^2 = \lambda \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n\right)$$

$\Rightarrow \lambda = 1.$

Now use: $\left(\tilde{a}_0 + \sum_{n=1}^{\infty} a_n q^n\right)\left(\tilde{b}_0 + \sum_{n=1}^{\infty} b_n q^n\right)$

$$= \tilde{a}_0 \tilde{b}_0 + (\tilde{a}_0 b_1 + a_1 \tilde{b}_0) q + \sum_{n=2}^{\infty} \left(\tilde{a}_0 b_n + a_n \tilde{b}_0 + \sum_{k=1}^{n-1} a_k b_{n-k}\right) q^n$$

We get:

$$\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^2 = 1 + (1 \cdot 240 + 240 \cdot 1) q$$

$$+ \sum_{n=2}^{\infty} \left(2 \cdot 240\, \sigma_3(n) + (240)^2 \sum_{k=1}^{n-1} \sigma_3(k) \sigma_3(n-k)\right) q^n$$

$$= 1 + 480 q + 480 \sum_{n=2}^{\infty} \left(\sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k) \sigma_3(n-k)\right) q^n$$

Hence $\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k) \sigma_3(n-k) \quad \forall n \geq 2.$

For instance: $n = 2$, $\sigma_7(2) = 1^7 + 2^7 = 1 + 128 = 129$

$\sigma_3(2) = 1^3 + 2^3 = 9$

$9 + 120\, \sigma_3(1) \cdot \sigma_3(1) = 129$ ✓

(4) We saw that G_2 is not modular. But, it can be used to construct modular forms.

First, define $E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = \frac{1}{2\zeta(2)} G_2 = \frac{3}{\pi^2} G_2$

We have $E_2(\tau+1) = E_2(\tau)$ (invariant under $\begin{pmatrix}1&1\\0&1\end{pmatrix}$)

but $E_2(-1/\tau) = \tau^2 E_2(\tau) - (2\pi i \tau) \cdot \frac{3}{\pi^2}$

$= \tau^2 E_2(\tau) - \frac{i6 \cdot \tau}{\pi}$ (almost inv. under $|_2 \begin{pmatrix}0&-1\\1&0\end{pmatrix}$)

If $f \in M_k(\Gamma)$ then $f(\tau+1) = f(\tau)$

$f(-1/\tau) = \tau^k f(\tau)$

thus $f'(\tau+1) = f(\tau)$ (invariant under $\begin{pmatrix}1&1\\0&1\end{pmatrix}$)

and $f'(-1/\tau) \cdot \frac{1}{\tau^2} = k\tau^{k-1} f(\tau) + \tau^k f'(\tau)$

$\Rightarrow \quad f'(-1/\tau) = \tau^{k+2} f'(\tau) + k\tau^{k+1} f(\tau)$ (almost inv. under $|_{k+2} \begin{pmatrix}0&-1\\1&0\end{pmatrix}$)

Using this one shows that $g(\tau) := \frac{1}{2\pi i} f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau)$

is a modular form in M_{k+2} (the extra terms cancel each other)

Rmk: • g is holomorphic in $\mathbb{H}$ because f, f' and E_2 are.

• The Fourier exp. of g is easy to compute. It has constant term $\frac{-k}{12} a_0$ where a_0 is the constant term of f

Example: $f = E_4 \Rightarrow g \in M_6 = \mathbb{C} E_6$ (see Lecture 6)

$a_0 = 1 \Rightarrow g = -\frac{4}{12} E_6 = -\frac{1}{3} E_6$

This implies identities beetwen Fourier coeffs (in this case involving $\sigma_3(n), \sigma_1(n), \sigma_5(n)$).