Exercise Class # 3 (31/March/2023)

We will review the exercises of the third sheet.

(1.a) Show that $E_4(\tau)$ and $E_6(\tau)$ are algebraically independent over $\mathbb{C}$, i.e. if $p(X,Y) \in \mathbb{C}[X,Y]$ w/ $p(E_4, E_6) = 0$, then $p(X,Y) = 0$.

Write $p(X,Y) = \sum_{a,b \geq 0} c_{a,b} X^a Y^b$. We want to show that $c_{a,b} = 0 \ \forall a,b \geq 0$ (integers).

Recall: $E_4 \in M_4, E_6 \in M_6 \Rightarrow E_4^a E_6^b \in M_{4a+6b}$

$$p(E_4, E_6) = \sum_{a,b \geq 0} \underbrace{c_{a,b} E_4^a E_6^b}_{\in M_{4a+6b}} = \sum_{k \geq 0}^{N} \underbrace{\left(\sum_{\substack{a,b \geq 0 \\ 4a+6b=k}} c_{a,b} E_4^a E_6^b \right)}_{=: f_k \in M_k}$$

for some $N \geq 0$ integer.

Here $f_k = p_k(E_4, E_6)$, $p_k(X,Y) = \sum_{\substack{a,b \geq 0 \\ 4a+6b=k}} c_{a,b} X^a Y^b$

Since $p = \sum_{k=0}^{N} p_k$, we have to show that $p_k = 0, \forall k$.

By hypothesis $\sum_{k \geq 0}^{N} f_k = p(E_4, E_6) = 0$.

Since the f_k have different weights, we have $f_k = 0 \ \forall k \geq 0$

Indeed:

$$f_0(\tau) + f_1(\tau) + \dots + f_N(\tau) = 0 \quad / \ \tau \mapsto -1/\tau$$
$$f_0(\tau) + \tau f_1(\tau) + \dots + \tau^N f_N(\tau) = 0 \quad / \ \tau \mapsto \tau+1$$
$$f_0(\tau) + (\tau+1) f_1(\tau) + \dots + (\tau+1)^N f_N(\tau) = 0 \quad / \ \tau \mapsto \tau+1$$
$$\vdots$$
$$f_0(\tau) + f(\tau+N-1) f_1(\tau) + \dots + (\tau+N-1)^N f_N(\tau) = 0$$

This implies

$$\underbrace{\begin{pmatrix} 1 & \tau & \cdots & \tau^N \\ 1 & (\tau+1) & \cdots & (\tau+1)^N \\ \vdots & & & \\ 1 & (\tau+N) & \cdots & (\tau+N)^N \end{pmatrix}}_{\text{This is a Vandermonde matrix}} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

* Vandermonde matrix: $M = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & & & & \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}$

has $\det(M) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i) \neq 0$ if $\alpha_j \neq \alpha_i$ $\forall j, i$ s.t. $j \neq i$.

Hence $f_0 = f_1 = \cdots = f_N = 0$.

We have shown that $f_k = P_k(E_4, E_6) = 0$. Now we prove that $P_k(X, Y) = 0$.

Let $U \subseteq \mathbb{H}$ be a small disc containing no zeroes of E_4 or E_6. In fact, by the valence formula (Lecture 6)

(*) $E_4(\tau) = 0 \iff \tau \equiv \rho := \frac{-1+i\sqrt{3}}{2} \mod SL_2(\mathbb{Z})$

$E_6(\tau) = 0 \iff \tau \equiv i \mod SL_2(\mathbb{Z})$

So we can choose any disc U contained in $\{\tau \in \mathbb{H} : \operatorname{Im}(\tau) > 1\}$

Then we can define $F := E_4^{1/2}$, $G := E_6^{1/3}$ using an analytic branch of log ($F = e^{\frac{1}{2}\log(E_4)}$, $G = e^{\frac{1}{3}\log(E_6)}$)

We have

$$\sum_{4a+6b=k} c_{a,b} F^{2a} G^{3b} = 0 \quad \Big/ \cdot \frac{1}{G^{k/2}} \qquad (k \text{ is even})$$

$$\sum_{4a+6b=k} c_{a,b} \left(\frac{F}{G}\right)^{2a} = 0$$

(*) $E_4(-1/\rho) = \rho^4 E_4(\rho)$ but $\frac{-1}{\rho} = \rho + 1$ and $E_4(\rho+1) = E_4(\rho)$. Since $\rho^4 = \rho + 1$, $E_4(\rho) = 0$. Similarly $E_6(i) = 0$.

Put $Q_k(X) = \sum_{4a+6b=k} c_{a,b} X^{2a}$. If $Q_k(X) \neq 0$, then

$\exists \lambda \in \mathbb{C}$ root of $Q_k(X)$ such that $\frac{F}{G} = \lambda$ in $\mho$

(since $\frac{F}{G}$ is holomorphic in $\mho$). This implies $E_4^3 = \lambda^6 \cdot E_6^2$.
Evaluating at $\tau = i$ we get $0 \neq E_4(i)^3 = \lambda^6 \cdot 0 = 0$. Hence
$Q_k(X) = 0$ so $c_{a,b} = 0$ $\forall a,b \geq 0$ integers.

Remark: To prove that E_4, E_6 are alg. indep. it is enough to prove that E_4^3, E_6^2 are alg. indep. In this case the proof simplifies a bit (no need to define F, G)

(1.b) Prove that $\{E_4^a E_6^b : 4a+6b=k\}$ is a basis of M_k.
By 1.a we know that the products $E_4^a E_6^b$ are linearly independent, hence it remains to prove that it has $\dim_{\mathbb{C}} M_k$ elements. This follows from

$$\#\{(a,b) \in \mathbb{Z}_0^+ : 4a+6b=k\} = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1, & \text{if } k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \pmod{12} \end{cases}$$

(prove it!)

$$= \dim_{\mathbb{C}} M_k \quad \text{(Lecture 6)}$$

(2) We want to prove $\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$

where $\Delta = \frac{E_4^3 - E_6^2}{1728}$

(2.a) Define $F(\tau) := q \prod_{n=1}^{\infty} (1-q^n)^{24}$. Prove $(\log F)' = 2\pi i E_2$

where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$

We compute

$$(\log F)' = \left(\log q + 24 \sum_{n=1}^{\infty} \log(1-q^n) \right)'$$

$$= \frac{q'}{q} + 24 \sum_{n=1}^{\infty} \frac{(-q^n)'}{1-q^n}$$

$q' = 2\pi i \cdot q \quad (q = e^{2\pi i \tau})$
$(q^n)' = 2\pi i n \cdot q$

$$= 2\pi i - (2\pi i)\, 24 \sum_{n=1}^{\infty} \frac{nq}{1-q^n}$$

$$= 2\pi i \left(1 - 24 \underbrace{\sum_{n=1}^{\infty} n \sum_{l=1}^{\infty} q^{nl}}_{N = ql} \right)$$

$\frac{r}{1-r} = \sum_{l=1}^{\infty} r^l$

$$\sum_{N=1}^{\infty} q^N \underbrace{\sum_{n \mid N} n}_{\sigma_1(N)}$$

$$= 2\pi i \; E_2(\tau)$$

Remark: In general $\log(z \cdot w) = \log(z) + \log(w) + \text{const.}$ w/ $e^{\text{const}} = 1$, but $(\text{const})' = 0$.

(2.b) We now prove that $F \in S_{12}$

We have $F(\tau+1) = F(\tau)$ since $q = e^{2\pi i \tau}$ is invariant under $\tau \mapsto \tau+1$. We have to prove

$$F(-1/\tau)\,\tau^{-12} = F(\tau) \qquad \left(\text{action of } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ in weight } 12\right)$$

Recall $E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau$ (Ex. class #2)

We get

$$\begin{aligned}
\operatorname{Log}\left(F(-1/\tau)\,\tau^{-12}\right)' &= \left(\operatorname{Log}(F(-1/\tau)) - 12\operatorname{Log}(\tau)\right)' \\
&= \operatorname{Log}(F)'\left(\tfrac{-1}{\tau}\right)\left(\tfrac{1}{\tau^2}\right) - 12\left(\tfrac{1}{\tau}\right) \\
&= 2\pi i\, E_2\left(\tfrac{-1}{\tau}\right)\tfrac{1}{\tau^2} - 12\left(\tfrac{1}{\tau}\right) \\
&= 2\pi i\left(\tau^2 E_2(\tau) - \tfrac{6i}{\pi}\tau\right)\tfrac{1}{\tau^2} - \tfrac{12}{\tau} \\
&= 2\pi i\, E_2(\tau) \\
&= \left(\operatorname{Log} F(\tau)\right)'
\end{aligned}$$

This implies $F(-1/\tau)\,\tau^{-12} = \lambda F(\tau)$ for some $\lambda \in \mathbb{C}$

Evaluating at $\tau = i$ we get $\lambda = 1$ since

$$F(i) = e^{-2\pi}\prod_{n=1}^{\infty}(1 - e^{-2\pi n})^{24} > 0.$$

This proves that $F \in M_{12}$. Since $\lim_{\operatorname{Im}(\tau)\to\infty} F(\tau) = 0$,

we get $F \in S_{12}$

(2.c) $F \in S_{12} = \mathbb{C}\Delta$. By comparing the first Fourier coeff we get $F = \Delta$

(3.a) The identity $E_{12} - E_6^2 = c\Delta$ w/ $c = \frac{2^6 3^5 7^2}{691}$ follows from direct computations using that $S_{12} = \mathbb{C}\Delta$

(3.b) Recall
$$E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n$$
$$E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n$$

hence $E_6^2 = 1 + \sum_{n \geq 1} a(n) q^n$, $a(n) \in \mathbb{Z}$, and

$$\frac{65520}{691} \sigma_{11}(n) + a(n) = \frac{2^6 3^5 \cdot 7^2}{691} \tau(n) \quad / \cdot 691$$
$$65520\, \sigma_{11}(n) + 691\, a(n) = 2^6 3^5 7^2 \tau(n)$$

$$\Rightarrow \quad 65520\, \sigma_{11}(n) \equiv 2^6 3^5 7^2 \tau(n) \mod 691.$$

Using that 691 is a prime that does not divide 65520 nor $2^6 3^5 7^2$, we get $\sigma_{11}(n) \equiv \tau(n) \mod 691$

(4a) Recall $j = \frac{E_4^3}{\Delta} = 1728 \frac{E_4^3}{E_4^3 - E_6^2}$.

Using $E_4(\rho) = 0$, $E_4(i) \neq 0$, $E_6(i) = 0$ one gets $j(\rho) = 0$ and $j(i) = 1728$

Now, $e^{2\pi i(-\bar{\tau})} = e^{\overline{2\pi i \tau}} = \overline{e^{2\pi i \tau}} = \bar{q}$, hence

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \Rightarrow E_4(-\bar{\tau}) = \overline{E_4(\tau)}$$

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \Rightarrow \Delta(-\bar{\tau}) = \overline{\Delta(\tau)}$$

This implies $j(-\bar{\tau}) = \overline{j(\tau)}$

(4.b) For $it \in C$ $(t > 1)$ we have $e^{2\pi i(it)} = e^{-2\pi t} \in]0,1[$

hence $E_4(it) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) e^{-2\pi t n} > 0$

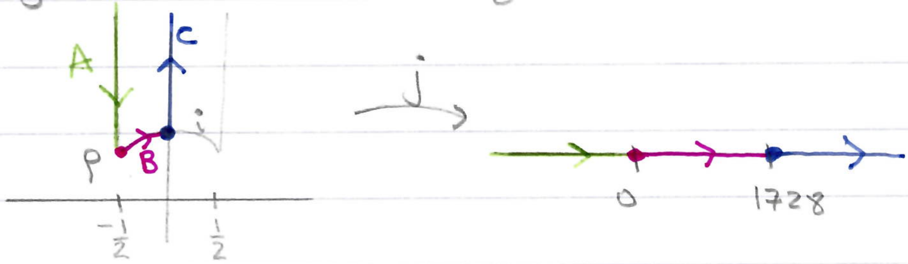
$$\Delta(it) = e^{-2\pi t} \prod_{n=1}^{\infty} \left(1 - e^{-2\pi t n}\right)^{24} > 0$$

So $j(it) > 0$. Moreover, $\lim_{t \to \infty} j(it) = \infty$, $j(i) = 1728$

and j restricted to C is injective, hence $j(C) =]1728, \infty[$.

In fact $j(it)$ is increasing for $t \geq 1$.

Similarly:



(For $\tau \in B$ use $\overline{j(\tau)} = j(-\overline{\tau}) = j\left(-\frac{1}{\tau}\right) = j(\tau)$ and for $\tau \in A$ use $\overline{j(\tau)} = j(-\overline{\tau}) = j(\tau+1) = j(\tau)$, so $j(B)$ and $j(A)$ are contained in $\mathbb{R}$)

From Lecture 7 we know $j: A \cup B \cup C \cup \{p, i\} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \to \mathbb{C}$ is bijection, hence $j(\mathcal{F}_1) \cup j(\mathcal{F}_2) = \mathbb{H} \cup \mathbb{H}^-$. Now, since j is holomorphic it preserves orientation, thus $j(\mathcal{F}_1) = \mathbb{H}$ and $j(\mathcal{F}_2) = \mathbb{H}^-$.