Exercise Class #4

28/April/2023

1 Prove the duplication formula

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s), \quad s\in\mathbb{C}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$ for $Re(s)>0$,

which extends to $s\in\mathbb{C}$ via $\Gamma(s+1)=s\Gamma(s)$

Remark $\Gamma(s+1)=s\Gamma(s)$ when $Re(s)>0$ follows from integration by parts. Indeed:

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^s dt = -\int_0^\infty (e^{-t})' t^s dt$$

$$= -e^{-t}\cdot t^s\Big|_0^\infty + \int_0^\infty e^{-t}(t^s)'\,dt$$

$$= 0 + \int_0^\infty e^{-t} s t^{s-1} dt$$

$$= s\,\Gamma(s)$$ //

1.a First prove $\Gamma(1/2)=\sqrt{\pi}$. To do this compute

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}dx\,dy = \left(\int_{-\infty}^\infty e^{-t^2}dt\right)^2 = \left(2\int_0^\infty e^{-t^2}dt\right)^2$$

$$(u=t^2) \qquad = \left(\int_0^\infty e^{-u} u^{-1/2} du\right)^2 = \Gamma(1/2)^2$$

and using polar coordinates this equals

$$\int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr = 2\pi\left(\frac{-e^{-r^2}}{2}\right)\Bigg|_{r=0}^{r=\infty} = \pi$$

. Hence $\Gamma(1/2)=\sqrt{\pi}$ because $\Gamma(1/2)>0$.

1.b Now prove

$$B(s_1, s_2) := \frac{\Gamma(s_1)\,\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}\,du$$

where $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 0$.

We have

$$\Gamma(s_1)\,\Gamma(s_2) = \int_0^\infty \int_0^\infty e^{-(x+y)}\, x^{s_1-1}\, y^{s_2-1}\,dx\,dy$$

Using $T :]0,\infty[\times]1,\infty[\longrightarrow]0,\infty[^2$

$$(u, v) \longmapsto (x,y) = (uv, u(1-v))$$

with $\left|\det\left(J_T(u,v)\right)\right| = \left|\det\begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}\right| = u$

gives $\Gamma(s_1)\,\Gamma(s_2) = \displaystyle\int_0^\infty \int_0^1 u e^{-u}\, u^{s_1+s_2-2}\, v^{s_1-1}(1-v)^{s_2-1}\,du\,dv$

$$= \Gamma(s_1+s_2)\int_0^1 v^{s_1-1}(1-v)^{s_2-1}\,dv \quad \text{as desired}$$

1.c Show $B(s,s) = 2^{2-2s}\displaystyle\int_0^1 (1-u^2)^{s-1}\,du = 2^{1-2s}B\left(\tfrac{1}{2}, s\right)$

and prove the duplication formula.

We have $B(s,s) = \displaystyle\int_0^1 v^{s-1}(1-v)^{s-1}\,dv \qquad v = \frac{1+x}{2}$

$$= \frac{1}{2}\int_{-1}^1 (1-x^2)^{s-1}\, 2^{2-2s}\,dx$$

$$= 2^{2-2s}\int_0^1 (1-x^2)^{s-1}\,dx \qquad t = x^2$$

$$= 2^{1-2s}\int_0^1 (1-t)^{s-1}\, t^{-1/2}$$

$$= 2^{1-2s}\, B\left(\tfrac{1}{2}, s\right)$$

Hence $\dfrac{\Gamma(s)}{\Gamma(2s)} = 2^{1-2s}\dfrac{\Gamma(1/2)\,\Gamma(s)}{\Gamma(s+1/2)}$

$$\Rightarrow \Gamma(s)\,\Gamma\left(s+\tfrac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s)$$

2 Let $k \geq 4$ integer, $L_k(s) := \zeta(s)\zeta(s-k+1)$, $s \in \mathbb{C}$, $\mathrm{Re}(s)$

2.a Show that $L_k(s)$ has mero. cont. to $s \in \mathbb{C}$ with

$$\Lambda_k(s) := (2\pi)^{-s}\Gamma(s)L_k(s) = (-1)^{k/2}\Lambda(k-s)$$

Rmk: This is the functional eq. of the L-function of a holomorphic mod. form of weight k.

We have

$$\Lambda_k(s) = (2\pi)^{-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)2^{s-1}\pi^{-1/2}\zeta(s)\zeta(s-k+1)$$

$$= \frac{1}{2}\underbrace{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)}_{\Lambda(s)}\ \underbrace{\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)\zeta(s-k+1)}_{\Lambda(s-k+1)\frac{\Gamma\left(\frac{s+1}{2}\right)}{\pi^{k/2}\Gamma\left(\frac{s-k+1}{2}\right)}}$$

$$= \frac{1}{2\pi^{k/2}}\Lambda(s)\Lambda(s-k+1)\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}-\frac{k}{2}\right)}$$

Using $\Gamma(s+1) = s\Gamma(s)$ we get

$$\Lambda_k(s) = \frac{1}{2\pi^{k/2}}\Lambda(s)\Lambda(s-k+1)\underbrace{\left(\frac{s+1}{2}-1\right)\left(\frac{s+1}{2}-2\right)\cdots\left(\frac{s+1}{2}-\frac{k}{2}\right)}_{p_k(s)}$$

and

$$p_k(k-s) = \left(\frac{k}{2}-\frac{s+1}{2}\right)\cdots\left(1-\frac{s+1}{2}\right) = (-1)^{k/2}p_k(s).$$

Since $\Lambda(s)$ has mero. cont. to $s \in \mathbb{C}$ we have the same for $\Lambda_k(s)$. Moreover

$$\Lambda_k(k-s) = \frac{1}{2\pi^{k/2}}\underbrace{\Lambda(k-s)}_{\Lambda(1-k+s)}\ \underbrace{\Lambda(1-s)}_{\Lambda(s)}(-1)^{k/2}p_k(s)$$

$$= (-1)^{k/2}\Lambda_k(s).$$

2.b $\Lambda_k(s)$ is holom. in $\mathbb{C}\setminus\{0,k\}$ w/ simple poles at $s=0,k$.

Clearly $\Lambda_k(s)$ is holom. in $\mathbb{C}\setminus\{0,1,k-1,k\}$

At $s=1$: $\Lambda(s-k+1)$ is holomorphic

$\Lambda(s)$ has simple pole

$p_k(s)$ vanishes

hence $\Lambda_k(s)$ is holom. at $s=1$

At $s=k-1$: $\Lambda(s-k+1)$ has simple pole

$\Lambda(s)$ is holomorphic

$p_k(s)$ vanishes

hence $\Lambda_k(s)$ is holom. at $s=k$

Note that $\deg(p_k)=k/2 \geq 2$ and $p_k(s)$ vanishes at all odd integers $s\in\{1,3,\ldots,k-1\}$.

At $s=0$: $\Lambda_k(s)$ has simple pole since $p_k(0)\neq 0$ and similarly at $s=k$.

2.c $L_k(s) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s}$

Indeed, $\zeta(s)\zeta(s-k+1) = \sum_{n\geq 1}\frac{1}{n^s}\sum_{m\geq 1}\frac{1}{m^s}m^{k-1}$

$$= \sum_{N=1}^{\infty}\sum_{m/N}\frac{m^{k-1}}{N^s} \qquad (N=nm)$$

$$= \sum_{N=1}^{\infty}\frac{\sigma_{k-1}(N)}{N^s} \quad //$$

Recall $E_k(s) = 1 + c_k\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$ with $c_k = \frac{-2k}{B_k}$

hence $L_k(s) = L\left(\frac{1}{c_k}E_k, s\right)$.

For $k=2$ the function $\Lambda_k(s)$ has poles at $s=0,1,2$ since $p_k(s)=\frac{s-1}{2}$ and $\Lambda(s)\Lambda(s-1)$ has a pole of order 2 at $s=1$

Hence $\Lambda_k(s)$ does not correspond to a holomorphic modular form.

3.a $(a_n)_{n\in\mathbb{N}}$ sequence in $\mathbb{C}$ w/ $a_1 \neq 0$ such that $\sum_n a_n$ conv. absolutely. Assume $(a_n)_n$ is multiplicative i.e. $a_{nm} = a_n a_m$ $\forall n,m$ coprime. Show

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 + a_p + a_{p^2} + \dots\right)$$

We have $a_1 = a_1 \cdot a_1$ hence $a_1 = 1$. Now, given $M \in \mathbb{Z}^+$ put

$$P_M := \{p \in \mathbb{Z}^+ \text{ prime}, \ p \leq M\}$$

$$I_M := \{n \in \mathbb{Z}^+ ; \text{ all prime divisors of } n \text{ are in } P_M\}$$

The series

$$S_p := \sum_{k=0}^{\infty} a_{p^k} = 1 + a_p + a_{p^2} + \dots$$

conv. abs. Since $(a_n)_{n\in\mathbb{N}}$ is multiplicative, we have

$$\prod_{p \in P_M} S_p = \sum_{n \in I_M} a_n$$

If q_M is the first prime $> M$, then

$$\left| \sum_{n \in I_M} a_n - \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n \geq q_M} a_n \xrightarrow[M \to \infty]{} 0$$

hence

$$\prod_p S_p = \sum_{n=1}^{\infty} a_n$$

Moreover, the convergence of the infinite product is absolute.

Absolute convergence of infinite products (Lang "Complex Analysis" Chapter XIII)

Let $(u_n)_{n\in\mathbb{N}}$ a seq. of non-zero complex numbers

Defn: The infinite product $\prod_{n=1}^{\infty} u_n$ converges absolutely if

(i) $\lim_{n\to\infty} u_n = 1$

(ii) $\sum_{n=1}^{\infty} \log(u_n)$ conv. absolutely (i.e. $\sum_{n=1}^{\infty} |\mathrm{Log}(u_n)| < \infty$)

where for $\log(u_n)$ we choose the principal branch of log ($\mathrm{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$ if $|z-1|<1$) provided $|u_n - 1| < 1$

this holds for n big enough because of (i)

Rmk: The condition $u_n \neq 0 \ \forall n$ is made to avoid trivial infinite products, since otherwise $\prod_{n=1}^{m} u_n = 0 \ \forall m$ big enough.

Lemma 1: abs. convergence $\Rightarrow$ usual convergence (i.e $\lim_{m\to\infty} \prod_{n=1}^{m} u_n$ exists)

Proof: $\sum_{n=1}^{\infty} \mathrm{Log}(u_n)$ conv abs $\Rightarrow \sum_{n=1}^{\infty} \mathrm{Log}(u_n)$ converges

$$\Rightarrow e^{\sum_{n=1}^{\infty} \mathrm{Log}(u_n)} \text{ converges}$$

$$= \prod_{n=1}^{\infty} e^{\mathrm{Log}(u_n)} \quad \text{by continuity of } e^z$$

$$= \prod_{n=1}^{\infty} u_n \ //$$

Lemma 2: $\sum_{n=1}^{\infty}(u_n - 1)$ conv. abs $\Rightarrow \prod_{n=1}^{\infty} u_n$ conv. absolutely.

Proof: If $\sum_{n=1}^{\infty}(u_n-1)$ conv. abs then $u_n - 1 \longrightarrow 0$

So $u_n \longrightarrow 1$. Hence (i) is satisfied.

Now, $\exists\, M \in \mathbb{N}$ such that $|u_n - 1| < \frac{1}{2}$ $\forall n \geq M$

Hence for $n \geq M$

$$\log(u_n) = \log(a_n + 1), \quad a_n := u_n - 1,\ |a_n| < 1/2$$
$$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} a_n^m$$

$$\Rightarrow |\log(u_n)| \leq |a_n| \cdot \underbrace{\sum_{m=1}^{\infty} \frac{1}{m\, 2^{m-1}}}_{C \,\leq\, \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \,<\, \infty} \leq C\,|a_n|$$

$$\Rightarrow \sum_{n \geq M} |\log(u_n)| \leq C \sum_{n \geq M} |a_n| = C \cdot \sum_{n \geq M} |u_n - 1| < \infty$$

by hypothesis

Hence (ii) is satisfied //

In our case $\prod_p (1 + a_p + a_{p^2} + \dots)$ conv. absolutely

because $\sum_p (a_p + a_{p^2} + \dots)$ conv abs

$$\left(\sum_p |a_p + a_{p^2} + \dots| \leq \sum_{n=2}^{\infty} |a_n| < \infty \right)$$

Hence we can use Lemma 2 //

3.b) Show that $L_k(s) = \prod_p \left((1 - p^{-s})(1 - p^{k-1-s}) \right)^{-1}$ for $s \in \mathbb{C}$ w/ $Re(s) > k$.

It is enough to prove $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for $s \in \mathbb{C}$ w/ $Re(s) > 1$. This follows from 3.a) with $a_n = \frac{1}{n^s}$

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_p \underbrace{\sum_{e=0}^{\infty} (p^{-s})^e}_{\frac{1}{1-p^{-s}}} = \prod_p (1 - p^{-s})^{-1} \quad //$$

4) $f_1 = \Delta^2$, $f_2 = \Delta \cdot E_6^2 \in S_{24}$, where

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

$$E_6(q) = 1 - 504q - 16632q^2 - 122976q^3 + \dots$$

4.a) Show $S_{24} = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

We know $\dim S_{24} = \left[\frac{24}{12}\right] = 2$

Clearly $f_1(q) = q^2 + \dots$, $f_2(q) = q + \dots$ so they are lin. indep., hence $S_{24} = \mathbb{C}f_1 \oplus \mathbb{C}f_2$

4.b) Find the matrix of T_2 in the basis $\mathcal{B} = \{f_1, f_2\}$

We have $T_2(f_1) = \sum_n c_n q^n$, $T_2(f_2) = \sum_n d_n q^n$

and $c_n = \sum_{t \mid (2,n)} t^{23} a_{\frac{2n}{t^2}}$ if $f_1 = \sum_n a_n q^n$

$d_n = \sum_{t \mid (2,n)} t^{23} b_{\frac{2n}{t^2}}$ if $f_2 = \sum_n b_n q^n$

In particular $c_1 = a_2$, $c_2 = a_4 + 2^{23} a_1$

$d_1 = b_2$, $d_2 = b_4 + 2^{23} b_1$

One computes

$$f_1 = q^2 - 48q^3 + 1080q^4 + \dots$$

$$f_2 = q - 1032q^2 + 245196q^3 + 10965568q^4 + \dots$$

which gives a_2, a_4, b_2, b_4. With this:

$c_1 = 1$, $c_2 = 1080$

$d_1 = -1032$, $d_2 = 19354176$

Now $T_2(f_1) = \alpha f_1 + \beta f_2 \Rightarrow \alpha = 2112$, $\beta = 1$

$T_2(f_2) = \gamma f_1 + \delta f_2 \Rightarrow \gamma = 18289152$, $\delta = -1032$

$$\Rightarrow [T_2]_{\mathcal{B}} = \begin{pmatrix} 2112 & 18289152 \\ 1 & -1032 \end{pmatrix}$$

4.c) Find a basis of S_{24} of normalized eigenforms for all T_n's.

The eigenvalues of $[T_2]_{\mathcal{B}}$ are (put $\omega := \sqrt{144169}$)

$\lambda_1 = 12(45+\omega)$ with vector $(12(131+\omega), 1)$

$\lambda_2 = 12(45-\omega)$ with vector $(12(131-\omega), 1)$

hence $F_1 := 12(131+\omega) f_1 + f_2 = q + 12(45+\omega)q^2 + \dots$

$F_2 := 12(131-\omega) f_1 + f_2 = q + 12(45-\omega)q^2 + \dots$

forms a basis $\{F_1, F_2\}$ of S_{24} of eigenforms for T_2. Since they correspond to different eigenvalues, it follows that they are also eigenforms for all T_n's (since T_n preserves eigenspaces of T_2) //