Exercise Class #5

1 Recall $E_k = \frac{1}{2\zeta(k)} G_k$ ($k \geq 4$ integer)

where $$G_k(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}, \quad z \in \mathbb{H}.$$

Now, $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subseteq \Gamma = SL_2(\mathbb{Z})$ acts by left multiplication on Γ as follows:

$$\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

and the map

$$\Gamma_\infty \backslash \Gamma \longrightarrow \{ (c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1 \} / \sim$$

where $(c,d) \sim (-c,-d)$

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c,d)$$

is a bijection. Hence:

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma = \sum_{\substack{(c,d) \in \mathbb{Z}^2/\sim \\ \gcd(c,d)=1}} (cz+d)^{-k}$$

Thus: $$E_k(z) = \frac{1}{2\zeta(k)} \sum_{d=1}^{\infty} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{d^k} \frac{1}{(mz+n)^k} = \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(mz+n)^k}$$

$$= \sum_{\substack{(m,n) \in \mathbb{Z}^2/\sim \\ \gcd(m,n)=1}} \frac{1}{(mz+n)^k}$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma$$

Let $g \in S_k$. We want to prove $\langle E_k, g \rangle = 0$

We have

$$\langle E_k, g \rangle = \int_F E_k(z)\, \overline{g(z)}\, y^k\, d\mu(z) \qquad (F \text{ fund. domain for } \Gamma) \quad d\mu = \frac{dx\,dy}{y^2}$$

$$= \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \int_F (cz+d)^{-k}\, \overline{\underbrace{g(z)}_{g|_k(z)}}\, y^k\, d\mu(z) \qquad y = \mathrm{Im}(z)$$

$$= \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \int_F |cz+d|^{-2k}\, \overline{g(\gamma z)}\, \mathrm{Im}(z)^k\, d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_F \overline{g(\gamma z)}\, \mathrm{Im}(\gamma z)^k\, d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma F} \overline{g(z)}\, \mathrm{Im}(z)^k\, d\mu(z)$$

$$= \int_{F_\infty} \overline{g(z)}\, \mathrm{Im}(z)^k\, d\mu(z), \quad F_\infty = \bigcup_{\gamma \in \Gamma_\infty \backslash \Gamma} \gamma F \text{ fund dom. for } \Gamma_\infty$$

$$= \int_0^\infty \int_0^1 \overline{g(z)}\, \mathrm{Im}(z)^k\, d\mu(z) \qquad (\text{taking } F_\infty = [0,1] \times]0,\infty[)$$

$$= \int_0^\infty y^{k-2} \left(\int_0^1 \overline{g(z)}\, dx \right) dy$$

But $g(z) = \sum_{n=1}^{\infty} a_n q^n$ hence

$$\int_0^1 \overline{g(z)}\, dx = \sum_{n=1}^{\infty} \overline{a}_n \int_0^1 e^{-2\pi i n x} e^{-2\pi n y}\, dx = \sum_{n=1}^{\infty} \overline{a}_n e^{-2\pi n y} \int_0^1 e^{-2\pi i n x}\, dx$$

where $\int_0^1 e^{-2\pi i n x}\, dx = \frac{1}{-2\pi i n} e^{-2\pi i n x} \Big|_{x=0}^{x=1} = 0$ if $n \neq 0$. Thus $\langle E_k, g \rangle = 0$.

This computations are valid since $g(z)$ is cuspidal (hence $g(z) = O(e^{-2\pi y})$ as $y \to \infty$ and also $a_n = O(n^{k/2})$)

$\boxed{2}$ If $f = E_k$ then by $\boxed{1}$ we have $\langle E_k, P_n \rangle = 0$ since $P_n \in S_k$. Since $a_n = \left(\frac{-2k}{B_k}\right)\sigma_{k-1}(n) \neq 0$ we have

$$0 = \langle E_k, P_k \rangle \neq \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \cdot a_n$$

Also, note that the proof of Theorem 5.4 in Lecture 11 is not valid when $f = E_k$. Indeed:

$$\langle f, P_n \rangle = \int_F f(z) \overline{\left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i m z} |_k \gamma \right)} y^k \, d\mu(z)$$

$$\overset{(?)}{=} \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \underbrace{\int_F \underbrace{f(z)}_{f(\gamma z)(cz+d)^{-k}} \overline{e^{2\pi i n \gamma(z)}} (c\bar{z}+d)^{-k} y^k \, d\mu(z)}$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_F f(\gamma z) \overline{e^{2\pi i m \gamma(z)}} \operatorname{Im}(\gamma z)^k \, d\mu(z)$$

$$= \int_{F_\infty} f(z) \overline{e^{2\pi i m z}} \operatorname{Im}(z)^k \, d\mu(z)$$

$$\overset{(??)}{=} \sum_{m=0}^{\infty} a_m \int_{F_\infty} e^{2\pi i m z} \overline{e^{2\pi i n z}} \operatorname{Im}(z)^k \, d\mu(z)$$

$$= \sum_{m=0}^{\infty} a_m \underbrace{\int_0^\infty e^{-2\pi(m+n)y} y^{k-2}}_{} \underbrace{\int_0^1 e^{2\pi i (m-n)x} \, dx}_{0 \text{ if } m \neq n} dy$$

$$= a_n \cdot \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \text{ if } m = n.$$

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Note that (?) and (??) hold if we have abs. convergence.

But (??) is valid only if $f \in S_k$ (because otherwise the series is not absolutely convergent). Indeed:

$$\sum_{m=0}^{\infty} |a_m| \int_{F_\infty} e^{-2\pi m y} e^{-2\pi n y} y^{k-2} dx\, dy$$

$$= \sum_{m=0}^{\infty} |a_m| \int_0^{\infty} e^{-2\pi(m+n)y} y^{k-2} dy$$

$$= \sum_{m=0}^{\infty} |a_m| \frac{\Gamma(k-1)}{(2\pi(m+n))^{k-1}} = \frac{\Gamma(k-1)}{(2\pi)^{k-1}} \sum_{m=0}^{\infty} \frac{|a_m|}{(m+n)^{k-1}}$$

If $f = E_k$ then $|a_m| \geq c \cdot m^{k-1}$ hence

$$\sum_{m=0}^{\infty} \frac{|a_m|}{(m+n)^{k-1}} \geq c \sum_{m=0}^{\infty} \left(\frac{m}{m+n}\right)^{k-1} = \infty$$

3 Given $\Gamma' \subseteq \Gamma = SL_2(\mathbb{Z})$ cong subgp, $f, g \in S_k(\Gamma')$

$$\langle f, g\rangle_{\Gamma'} := \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]} \int_{\Gamma' \backslash \mathbb{H}} f(z)\,\overline{g(z)}\, y^k \, d\mu(z)$$

where $\overline{\Gamma'}$ is the image of Γ' under $\begin{array}{ccc} \Gamma & \longrightarrow & \overline{\Gamma} = PSL_2(\mathbb{Z}) \\ \gamma & \longmapsto & \overline{\gamma} = \pm\gamma \end{array}$

a) If f, g also belong to $S_k(\Gamma'')$, $\Gamma'' \subseteq \Gamma$ cong. subgp, we want to show that

$$\langle f, g\rangle_{\Gamma'} = \langle f, g\rangle_{\Gamma''}$$

Note that $f, g \in S_k(\Gamma' \cap \Gamma'')$ and $\Gamma' \cap \Gamma''$ is also a cong. subgp $\left(\Gamma' \supseteq \Gamma(N), \Gamma'' \supseteq \Gamma(M) \Rightarrow \Gamma' \cap \Gamma'' \supseteq \underbrace{\Gamma(N) \cap \Gamma(M)}_{= \Gamma(\mathrm{lcm}(N,M))} \supseteq \Gamma(NM) \right)$

It is enough to prove $\langle f, g\rangle_{\Gamma'} = \langle f, g\rangle_{\Gamma' \cap \Gamma''}$ so we can assume $\Gamma'' \subseteq \Gamma'$.

Let F' be a fund. domain for Γ'

Recall that if $\overline{\Gamma'} = \bigsqcup_{i=1}^{m} \overline{\Gamma''} \cdot \overline{\gamma_i}$ $\left(\begin{array}{c} m = [\overline{\Gamma'} : \overline{\Gamma''}] \\ \gamma_1, \dots, \gamma_m \in \Gamma' \end{array} \right)$

then $F'' := \bigcup_{i=1}^{m} \gamma_i F'$ is a fund domain for Γ''

with $\mu(\gamma_i F' \cap \gamma_j F') = 0$ if $i \neq j$ (Exercise Class #1)

Hence

$$\langle f, g\rangle_{\Gamma''} = \frac{1}{[\overline{\Gamma} : \overline{\Gamma''}]} \int_{F''} f(z)\,\overline{g(z)}\, y^k \, d\mu(z)$$

$$= \left(\quad '' \quad \right) \sum_{i=1}^{m} \int_{\gamma_i F'} \underbrace{f(z)\,\overline{g(z)}\, y^k}_{\Gamma'\text{-inv because } f,g \in S_k(\Gamma')} \, d\mu(z)$$

$$= \left(\frac{1}{[\overline{\Gamma} : \overline{\Gamma''}]} \right) \cdot m \int_{F'} f(z)\,\overline{g(z)}\, y^k \, d\mu(z)$$

$$= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]\cancel{m}} \cdot \cancel{m} \int_{F'} f(z)\,\overline{g(z)}\, y^k \, d\mu(z) = \langle f, g\rangle_{\Gamma'} \quad //$$

(b) Given $\alpha \in GL_2^+(\mathbb{Q})$ we will show that $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$ is a cong. subgp of $\Gamma = SL_2(\mathbb{Z})$

$\left(f \in S_k = S_k(\Gamma) \Rightarrow f|_k\alpha \in S_k(\Gamma') \right)$

Write $\alpha = \frac{1}{m}\beta$, $m \geq 1$ integer, $\beta \in M_{2\times 2}(\mathbb{Z})$.

Then $\alpha^{-1}\Gamma\alpha = \beta^{-1}\Gamma\beta$, so we can assume $\alpha \in M_{2\times 2}^+(\mathbb{Z})$

Let $D := \det(\alpha) \in \mathbb{Z}^+$. We will show that $\alpha^{-1}\Gamma\alpha \supseteq \Gamma(D)$, or equivalently $\alpha\Gamma(D)\alpha^{-1} \subseteq \Gamma$. For this, let $\gamma \in \Gamma(D)$.

Then $\gamma = I + DM$ where $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $M \in M_{2\times 2}(\mathbb{Z})$

$\Rightarrow \alpha\gamma\alpha^{-1} = I + \alpha M \underbrace{(D\alpha^{-1})}_{\in M_{2\times 2}(\mathbb{Z})} \in M_{2\times 2}(\mathbb{Z})$ Since $\det(\alpha\gamma\alpha^{-1}) = \det(\gamma) = 1$ we have $\alpha\gamma\alpha^{-1} \in \Gamma$, so $\alpha\Gamma(D)\alpha^{-1} \subseteq \Gamma$. This implies $\Gamma' \supseteq \Gamma(D)$ hence Γ' is a cong. subgp

(c) If $f \in S_k$ then $(f|_k\alpha)|_k(\alpha^{-1}\gamma\alpha) = (f|_k\gamma)|_k\alpha$
$= f|_k\alpha \quad \forall \gamma \in \Gamma$

hence $(f|_k\alpha)|_k\beta = (f|_k\alpha) \quad \forall \beta \in \Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$

The fact that $f|_k$ is holom. in $\mathbb{H}$ follows from the fact that f is holom. in $\mathbb{H}$. Finally, for every cusp representative $x \in \mathbb{Q}\cup\{\infty\}$ we have to prove that $(f|_k\alpha)$ vanishes at x. This means that if we choose $\sigma \in \Gamma$ with $\sigma(\infty) = x$, then $(f|_k\alpha)|_k\sigma$ vanishes at ∞. For this, choose $\gamma \in \Gamma$ such that $(\gamma\alpha\sigma)(\infty) = \infty$ (i.e. $\alpha(x) \in \mathbb{Q}\cup\{\infty\}$ and γ is such that $\gamma(\alpha(x)) = \infty$). Then $\gamma\alpha\sigma = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \in GL_2^+(\mathbb{Q})$

(or even in $M_{2\times 2}^+(\mathbb{Z})$ if $\alpha \in M_{2\times 2}^+(\mathbb{Z})$) and

$(f|_k\alpha\sigma) = f|_k\gamma\alpha\sigma = (ps)^{k/2} s^{-k} f\left(\frac{pz+q}{s}\right) \to 0$ as $\mathrm{Im}(z) \to \infty$.

so $(f|_k\alpha\sigma)$ vanishes at ∞. This proves that $f|_k\alpha \in S_k(\Gamma')$.

Now we show that

$$\langle f, g\rangle_{\Gamma} = \langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'} \quad \forall f, g \in S_k(\Gamma)$$

We have $\langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'} = \dfrac{1}{[\overline{\Gamma}:\overline{\Gamma'}]} \displaystyle\int_{F'} (f|_k\alpha)(z)\overline{(g|_k\alpha)(z)}\, y^k\, d\mu(z)$

(F' = fund domain for Γ')

$$= \frac{1}{[\overline{\Gamma}:\overline{\Gamma'}]} \int_{F'} f(\alpha z)\overline{g(\alpha z)}\, \mathrm{Im}(\alpha z)^k\, d\mu(z)$$

$$= \frac{1}{[\overline{\Gamma}:\overline{\Gamma'}]} \int_{\alpha F'} f(z)\overline{g(z)}\, \mathrm{Im}(z)^k\, d\mu(z)$$

The set $\alpha F'$ is a fund domain for $\Gamma'' := \alpha\Gamma'\alpha^{-1} = \alpha\Gamma\alpha^{-1}\cap\Gamma$, hence

$$\langle f, g\rangle_{\Gamma} \underset{\text{by (a)}}{=} \langle f, g\rangle_{\Gamma''} = \frac{1}{[\overline{\Gamma}:\overline{\Gamma''}]} \int_{\alpha F'} f(z)\overline{g(z)}\, y^k\, d\mu(z)$$

$$= \frac{[\overline{\Gamma}:\overline{\Gamma'}]}{[\overline{\Gamma}:\overline{\Gamma''}]} \langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'}$$

We just need to prove that $[\overline{\Gamma}:\overline{\Gamma'}] = [\overline{\Gamma}:\overline{\Gamma''}]$. For this, note that if F, F', F'' are fund domains for Γ, Γ' and Γ'', then

$$\mu(F') = [\overline{\Gamma}:\overline{\Gamma'}]\mu(F), \quad \mu(F'') = [\overline{\Gamma}:\overline{\Gamma''}]\mu(F)$$

Since we can take $F'' = \alpha F'$ and $\mu(F'') = \mu(\alpha F') = \mu(F')$ we get $[\overline{\Gamma}:\overline{\Gamma'}] = [\overline{\Gamma}:\overline{\Gamma''}]$ as desired //

d) We now prove $\langle T_n(f), g\rangle_\Gamma = \langle f, T_n(g)\rangle_\Gamma \quad \forall f,g \in S_k(\Gamma)$

Recall $T_n(f) = n^{k/2-1} \sum_{\gamma \in R(n)} f|_k\gamma$

where $R(n) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2^+(\mathbb{Z}) : ad=n,\ b \in Z_d \right\}$

with Z_d any set of reps for $\mathbb{Z}/d\mathbb{Z}$ ($R(n)$ is a set of reps for $\Gamma \backslash M(n)$)

Now, by c) we have $\langle f|_k\alpha, g\rangle = \langle f, g|_k\alpha^{-1}\rangle$ $\forall \alpha \in GL_2^+(\mathbb{Q})$. Here $\alpha^{-1} = \frac{1}{\det(\alpha)}\alpha'$ where for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

we write $\alpha' := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Since $g|_k\alpha^{-1} = g|_k\alpha'$ we get $\langle f|_k\alpha, g\rangle = \langle f, g|_k\alpha'\rangle$ $\forall \alpha \in GL_2^+(\mathbb{Q})$

Now

$$\langle T_n(f), g\rangle = n^{\frac{k}{2}-1} \sum_{\gamma \in R(n)} \langle f|_k\gamma, g\rangle$$

$$= n^{\frac{k}{2}-1} \sum_{\gamma \in R(n)} \langle f, g|_k\gamma'\rangle$$

But $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$ so $R(n)'$ gives another set of reps for $\Gamma \backslash M(n)$, so

$$\langle T_n(f), g\rangle = \langle f, T_n(g)\rangle \,_{//}$$

$\boxed{4}$ Let $k \geq 12$, $k \neq 14$ be an even integer

Let $a, b \geq 0$ integers s.t. $4a+6b \neq 12$, $4a+6b \leq 14$ and $4a+6b \equiv k \mod 12$ $\left((a,b) \in \{(0,0),(1,0),(0,1),(2,0),(1,1),(2,1)\} \right)$

Put $d := \dim S_k = \begin{cases} \lfloor k/12 \rfloor - 1 & \text{if } k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor & \text{if } k \not\equiv 2 \pmod{12} \end{cases}$

For $j \in \{1, \dots, d\}$ define $f_j = \Delta^j E_6^{2(d-j)+b} E_4^a$

(a) First, show that $f_j \in S_k$

The wt of f_j is $12j + 6(2(d-j)+b) + 4a = 12d + 6b + 4a$

If $k \equiv 2 \mod 12$ then $4a+6b = 14$. Also $d = \lfloor k/12 \rfloor - 1$

$k = 12(d+1) + 2 = 12d + 14 = 12d + 4a + 6b$, hence f_j has wt k

The fact that $f_j \in M_k$ is clear. It is cuspidal because its Fourier expansion is of the form

(☆) $f_j = q^j + O(q^{j+1})$ and $j \geq 1$.

Recall $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$, $\tau(n) \in \mathbb{Z}$

$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$

$E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$

also have integer ~~Fourier coeffs~~

hence f_j also has integer Fourier coeffs. So, we can write

$f_j = \sum_{n=1}^{\infty} a_n^{(j)} q^n$, $a_n^{(j)} \in \mathbb{Z}$ $\forall n$ $\forall j$

and by (☆) $a_n^{(j)} = 0$ if $1 \leq n < j$, $a_n^{(j)} = 1$ if $n = j$

(b) If $b_1 f_1 + \dots + b_d f_d = 0$ with $b_1, \dots, b_d \in \mathbb{C}$, then looking at the firsts Fourier coeffs we get $b_1 = 0, b_2 = 0, \dots, b_d = 0$

So $\{f_1, \dots, f_d\}$ is lin. indep. Since $d = \dim S_k$, it is a basis

(c) Any $g \in \mathbb{Z} f_1 + \dots + \mathbb{Z} f_d$ has integer Fourier coeffs. Conversely, if $g = \sum_{n=1}^{\infty} c_n q^n \in S_K$ with $c_n \in \mathbb{Z}, \forall n \geq 1$ then writing

$$g = \alpha_1 f_1 + \dots + \alpha_d f_d , \quad \alpha_i \in \mathbb{C}$$

we get

$$c_1 = \alpha_1 \cdot 1 \Rightarrow \alpha_1 \in \mathbb{Z}$$

$$c_2 = \alpha_1 a_2^{(1)} + \alpha_2 \cdot 1 \Rightarrow \alpha_2 \in \mathbb{Z}$$

$$\vdots$$

$$c_d = \alpha_1 a_d^{(1)} + \alpha_2 a_d^{(2)} + \dots + \alpha_d \cdot 1 \Rightarrow \alpha_d \in \mathbb{Z}$$

(by induction). Hence $g \in \mathbb{Z} f_1 + \dots + \mathbb{Z} f_d$