Exercise Sheet 1

1. **a.** For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with $c \neq 0$ and $z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$ show that

$$\mathrm{Im}\,(\gamma \circ z) = \frac{\det(\gamma) \cdot \mathrm{Im}(z)}{|cz+d|^2}, \quad \text{where } \gamma \circ z := \frac{az+b}{cz+d}.$$

b. Determine the image of the complex upper half-plane $\mathbb{H}$ under the action of γ.

c. Does $(\gamma, z) \mapsto \gamma \circ z$ define an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}$?

2. Let $\mathcal{L}$ be the set of all lattices in $\mathbb{C}$, where $\mathbb{C}$ is seen as a vector space over $\mathbb{R}$, and consider the action of $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ on $\mathcal{L}$ given by scalar multiplication. Given a function $F : \mathcal{L} \to \mathbb{C}$ define $f_F : \mathbb{H} \to \mathbb{C}$ by

$$f_F(\tau) := F(\tau\mathbb{Z} + \mathbb{Z}).$$

a. For $k \in \mathbb{Z}$ show that if $F : \mathcal{L} \to \mathbb{C}$ satisfies

$$F(\lambda L) = \lambda^{-k} F(L), \ \text{ for all } L \in \mathcal{L} \text{ and } \lambda \in \mathbb{C}^\times, \tag{1}$$

then f_F satisfies

$$f_F\left(\gamma \circ \tau\right) = (c\tau + d)^k f_F(\tau), \ \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \tau \in \mathbb{H}. \tag{2}$$

b. Show that $F \mapsto f_F$ defines a bijection between the set of all functions $F : \mathcal{L} \to \mathbb{C}$ satisfying (1) and the set of all functions $f : \mathbb{H} \to \mathbb{C}$ satisfying (2).

3. Parabolas, ellipses (circles being special cases of ellipses) and hyperbolas are called *conic sections* since they can be obtained from a cone's surface intersecting a plane. Algebraically, they can be realized as *quadratic curves* of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{3}$$

with $A, B, C, D, E, F \in \mathbb{R}$, A, B, C not all zero. The corresponding conic section is an ellipse (resp. a parabola, a hyperbola) if $B^2 - 4AC < 0$ (resp. $B^2 - 4AC = 0$, $B^2 - 4AC > 0$). Note, however, that a general quadratic equation (3) may have no real solutions, represent a single point, two intersecting lines, a pair of parallel lines, or one single line (of multiplicity two).[1]

Now, recall that a matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ different from $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called elliptic (resp. parabolic, hyperbolic) if $|\mathrm{tr}(\gamma)| < 2$ (resp. $|\mathrm{tr}(\gamma)| = 2$, $|\mathrm{tr}(\gamma)| > 2$).

a. Show that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is elliptic (resp. hyperbolic) if and only if the linear action of γ on $\mathbb{R}^2$ given by

$$\gamma \circ (x, y) := (ax + by, cx + dy),$$

leaves infinitely many ellipses (resp. hyperbolas) invariant. (An invariant subset of a map $f : X \mapsto X$ is a subset $Y \subseteq X$ such that $f(Y) \subseteq Y$.)

b. What are the invariant subsets of $\mathbb{R}^2$ under a parabolic transformation?

4. Write the matrix $\begin{pmatrix} 4 & -9 \\ -11 & 25 \end{pmatrix}$ in terms of $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

5. Consider the group

$$\Gamma_\theta \ := \ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2 \right\}.$$

a. Compute the group index of Γ_θ in $\mathrm{SL}_2(\mathbb{Z})$.

b. Show that $F_\theta := \{\tau \in \mathbb{H} : |\tau| \geq 1, |\mathrm{Re}(\tau)| \leq 1\}$ is a fundamental domain for Γ_θ.

c. Show that Γ_θ is generated by S and T^2.

[1] See the book *Analytic Geometry* by A. C. Bourdette (Academic Press 1971), chapters 6 and 7.