Exercise Sheet 2

1. The purpose of this exercise is to prove the identity

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{z+n} - \frac{1}{n}\right) \quad \text{for } z\in\mathbb{C}\setminus\mathbb{Z}, \tag{1}$$

and to deduce from it the formula

$$\zeta(k) = -\frac{(2\pi i)^k}{2(k!)}B_k \quad \text{for } k\in\mathbb{Z}^+ \text{ even}, \tag{2}$$

where the k-th Bernoulli number B_k is defined by

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}z^k. \tag{3}$$

a. Start by proving the identity

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) \quad \text{for } z\in\mathbb{C}\setminus\{0\}. \tag{4}$$

Hint: Use that

$$\frac{\sin(z)}{z} = \frac{e^{iz}-e^{-iz}}{2iz} = \lim_{n\to\infty} f_n(z)$$

where

$$f_n(z) := \frac{\left(1+\frac{iz}{n}\right)^n - \left(1-\frac{iz}{n}\right)^n}{2iz},$$

and show that

$$f_n(z) = \prod_{k=1}^{m}\left(1-\frac{z^2}{n^2}\left(\frac{1+\cos\left(\frac{2k\pi}{n}\right)}{1-\cos\left(\frac{2k\pi}{n}\right)}\right)\right) \quad \text{whenever } n=2m+1, m\in\mathbb{Z}^+.$$

b. Take logarithmic derivatives[1] on both sides of (4) to deduce (1).

c. Use (1) to prove

$$\pi z\cot(\pi z) = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)z^{2k} \quad \text{for } z\in\mathbb{C} \text{ with } |z|<1.$$

d. Find an alternative Taylor expansion for $\pi z\cot(\pi z)$ around 0, using (3), and obtain (2) by comparing Taylor coefficients.

2. The purpose of this exercise is to prove that the Eisenstein series $G_2:\mathbb{H}\to\mathbb{C}$ defined by

$$G_2(\tau) := \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

where $\sigma_s(n) := \sum_{d|n,d>0} d^s$ and $q := e^{2\pi i\tau}$, satisfies

$$G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) - 2\pi i\tau.$$

[1] Given a differentiable function $f(z)$ its logarithmic derivative at a point z with $f(z)\neq 0$ is defined as $\log(f(z))' = \dfrac{f'(z)}{f(z)}$.

In order to do this, consider the functions

$$\begin{aligned} F_1(\tau) &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{(m+n\tau)^2}, \\ F_2(\tau) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}}{}' \frac{1}{(m+n\tau)^2}, \\ H_1(\tau) &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{(m-1+n\tau)(m+n\tau)}, \\ H_2(\tau) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}}{}' \frac{1}{(m-1+n\tau)(m+n\tau)}, \end{aligned}$$

where the sign $\sum'$ indicates that (m,n) runs through all $m\in\mathbb{Z}, n\in\mathbb{Z}$ with $(m,n)\neq(0,0)$ for F_1 and F_2, and $(m,n)\neq(0,0),(1,0)$ for H_1 and H_2 (mind the order of the summations).

a. Show that

$$H_1(\tau) = 2 \text{ and } H_2(\tau) = 2 - \frac{2\pi i}{\tau},$$

and

$$F_1(\tau) - H_1(\tau) = F_2(\tau) - H_2(\tau).$$

b. Use **a.** to prove that

$$F_1(\tau) - F_2(\tau) = H_1(\tau) - H_2(\tau) = \frac{2\pi i}{\tau},$$

and deduce the identity

$$F_1\left(-\frac{1}{\tau}\right) = \tau^2 F_1(\tau) - 2\pi i\tau.$$

c. Prove that $F_1(\tau) = G_2(\tau)$.

3. Show that the normalized Eisenstein series $E_4, E_6, E_8, E_{10}, E_{14}$ (see Lecture 5) satisfy the relations $E_4^2 = E_8, E_4E_6 = E_{10}$ and $E_6E_8 = E_{14}$, and use these to derive relations between the arithmetic functions $\sigma_3(n), \sigma_5(n), \sigma_7(n), \sigma_9(n)$ and $\sigma_{13}(n)$.

4. Define the normalized Eisenstein series of weight 2 as

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

a. Given f in the space M_k of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ define

$$g(\tau) = \frac{1}{2\pi i} f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau),$$

where $f' = \dfrac{\partial f}{\partial \tau}$. Prove that $g \in M_{k+2}$ and that g is cuspidal if and only f is cuspidal.

b. Compute g explicitly when $f = E_4, E_6$ or Δ and derive relations between the arithmetic functions $\sigma_3(n), \sigma_5(n), \sigma_7(n)$ and $\tau(n)$ (the n-th Fourier coefficient of Δ).