Exercise Sheet 3

1. **a.** Show that the normalized Eisenstein series E_4 and E_6 are algebraically independent over $\mathbb{C}$. This is, if $p(X,Y) \in \mathbb{C}[X,Y]$ satisfies $p(E_4, E_6) = 0$, then $p(X,Y) = 0$.

b. Show that for every integer k the set

$$\{E_4^a E_6^b : a, b \in \mathbb{Z}_0^+, 4a + 6b = k\}$$

is a basis of M_k.

2. The purpose of this exercise is to show that the discriminant modular form, defined as

$$\Delta(\tau) := \frac{E_4^3(\tau) - E_6^2(\tau)}{1728} \quad \text{for } \tau \in \mathbb{H},$$

admits the infinite product expansion

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad \text{where } q := e^{2\pi i \tau}. \tag{1}$$

a. Let $F : \mathbb{H} \to \mathbb{C}$ be the function defined by the right hand side of (1). Show that the logarithmic derivative of F equals $2\pi i E_2$.

b. Use the transformation properties of E_2 with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ to show that F belongs to the space S_{12} of cuspidal modular forms of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

c. Use dimension formulas to conclude that $F = \Delta$.

3. **a.** Show that $E_{12} - E_6^2 = c\Delta$ where $c = \dfrac{2^6\, 3^5\, 7^2}{691}$. Use this to derive an expression for $\tau(n)$ (the n-th Fourier coefficient of Δ) in terms of σ_{11} and σ_5.

b. Prove Ramanujan's congruence relation

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \quad \text{for all } n \in \mathbb{Z}^+.$$

4. Show that the j modular function (see Lecture 7) satisfies the following properties.

a. $j\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 0$, $j(i) = 1728$ and $j(-\overline{\tau}) = \overline{j(\tau)}$ for all $\tau \in \mathbb{H}$.

b. The image under j of the sets

$$\begin{aligned} A &:= \left\{-\frac{1}{2} + it : t \in \left]\frac{\sqrt{3}}{2}, \infty\right[\right\}, \\ B &:= \left\{e^{it} : t \in \left]\frac{\pi}{2}, \frac{2\pi}{3}\right[\right\}, \\ C &:= \{it : t \in \,]1, \infty[\}, \end{aligned}$$

are $j(A) =]-\infty, 0[$, $j(B) =]0, 1728[$ and $j(C) =]1728, \infty[$.

c. The image under j of the sets

$$\begin{aligned} \mathcal{F}_1 &:= \left\{\tau \in \mathbb{H} : -\frac{1}{2} < \mathrm{Re}(\tau) < 0, |z| > 1\right\}, \\ \mathcal{F}_2 &:= \left\{\tau \in \mathbb{H} : 0 < \mathrm{Re}(\tau) < \frac{1}{2}, |z| > 1\right\}, \end{aligned}$$

are $j(\mathcal{F}_1) = \mathbb{H}$ and $j(\mathcal{F}_2) = \mathbb{H}^-$, where $\mathbb{H}^-$ denotes the complex lower half-plane.