Exercise Sheet 4

1. Let $\Gamma(s)$ denote Euler Gamma function (see Lecture 9). The purpose of this exercise is to prove the duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s). \tag{1}$$

a. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by computing

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$$

in two different ways.

b. For $s_1, s_2 \in H := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ show that

$$B(s_1, s_2) := \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}du.$$

c. For $s \in H$ show that

$$B(s,s) = 2^{2-2s}\int_0^1 (1-u^2)^{s-1}du = 2^{1-2s}B\left(\frac{1}{2}, s\right),$$

and deduce from this the desired duplication formula (1).

2. Given an even integer $k \geq 4$ define

$$L_k(s) := \zeta(s)\zeta(s-k+1) \text{ for } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > k. \tag{2}$$

a. Use the meromorphic continuation and functional equation of $\zeta(s)$ to show that $L_k(s)$ has meromorphic continuation to $\mathbb{C}$ satisfying the functional equation

$$\Lambda_k(s) := (2\pi)^{-s}\Gamma(s)L_k(s) = (-1)^{k/2}\Lambda_k(k-s).$$

(Hint: use the duplication formula (1) and the identity $\Gamma(s+1) = s\Gamma(s)$.)

b. Show that $\Lambda_k(s)$ is holomorphic in $\mathbb{C} \setminus \{0, k\}$ and has simple poles at $s = 0$ and $s = k$.

c. Show that

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s}$$

and conclude that $L_k(s)$ is the L-function of a multiple of the normalized Eisenstein series E_k. What happens if $k = 2$?

3. **a.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $a_1 \neq 0$ and such that $\sum_n a_n$ converges absolutely. Assume that the sequence $(a_n)_{n\in\mathbb{N}}$ is multiplicative, i.e. $a_{nm} = a_n a_m$ for all positive integers n, m with g.c.d.$(n, m) = 1$. Show that

$$\sum_{n=1}^{\infty} a_n = \prod_p \left(1 + a_p + a_{p^2} + \ldots\right)$$

where the product is taken over all primes and the convergence is absolute.

b. Show that the function $L_k(s)$ defined by (2) admits the infinite product representation

$$L_k(s) = \prod_p \left((1-p^{-s})(1-p^{k-1-s})\right)^{-1} \text{ for } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > k.$$

4. Let $f_1 := \Delta^2$ and $f_2 := \Delta E_6^2$, where Δ is the discriminant modular form and E_6 is the normalized Eisenstein series of weight 6.

a. Show that $\{f_1, f_2\}$ is a basis for S_{24}.

b. With the help of a calculator, find the matrix of T_2 in the basis $\{f_1, f_2\}$.

c. Express in terms of f_1 and f_2 the basis for S_{24} consisting of normalized eigenforms for all Hecke operators T_n.