Exercise Sheet 5

1. Let $k \geq 4$ be an even integer. Show that the normalized Eisenstein series E_k can be written as

$$E_k = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma,$$

where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$. Use this to prove that for every cusp form $g \in S_k$ we have $\langle E_k, g \rangle = 0$.

2. Let $k > 2$ and $n \geq 1$ be integers and let $P_n \in S_k$ be the n-th Poincaré series defined as

$$P_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z}|_k \gamma.$$

According to Theorem 5.4 in Lecture 11 for every cusp form $f \in S_k$ with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n q^n$ we have

$$\langle f, P_n \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_n.$$

Is this formula valid when f is an Eisenstein series? If yes, prove it. If not, explain why.

3. Given a congruence subgroup $\Gamma' \subseteq \Gamma = \mathrm{SL}_2(\mathbb{Z})$ denote by $\overline{\Gamma'}$ its image in $\overline{\Gamma} := \mathrm{PSL}_2(\mathbb{Z})$, and given two cusp forms $f, g \in S_k(\Gamma')$ define

$$\langle f, g \rangle_{\Gamma'} := \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]} \int_{\Gamma' \backslash \mathbb{H}} f(z)\overline{g(z)} \mathrm{Im}(z)^k d\mu(z)$$

where $d\mu$ is the hyperbolic measure and the integral over $\Gamma' \backslash \mathbb{H}$ is defined as the integral over any fundamental domain F' for Γ'.

a. Show that if Γ'' is another congruence subgroup of Γ such that $f, g \in S_k(\Gamma'')$, then

$$\langle f, g \rangle_{\Gamma''} = \langle f, g \rangle_{\Gamma'}.$$

b. Given $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ show that $\Gamma' := \Gamma \cap \alpha^{-1}\Gamma\alpha$ is a congruence subgroup of Γ.

c. Show that for any $f, g \in S_k(\Gamma)$ we have $f|_k\alpha, g|_k\alpha \in S_k(\Gamma')$ and

$$\langle f|_k\alpha, g|_k\alpha \rangle_{\Gamma'} = \langle f, g \rangle_{\Gamma}.$$

d. Use **c.** to give an alternative proof of the fact that Hecke operators on $S_k(\Gamma)$ are self-adjoint with respect to the Petersson inner product.

4. Let $14 \neq k \geq 12$ be an even integer and let $a, b \geq 0$ be integers such that $12 \neq 4a + 6b \leq 14$ and $4a + 6b \equiv k \pmod{12}$. Let d be the dimension of S_k and for each $j \in \{1, \ldots, d\}$ define

$$f_j := \Delta^j E_6^{2(d-j)+b} E_4^a.$$

a. Show that $f_j \in S_k$ and has Fourier expansion $f_j(z) = \sum_{n=1}^{\infty} a_n^{(j)} q^n$ satisfying $a_n^{(j)} \in \mathbb{Z}$ for all n. Moreover, show that $a_n^{(j)} = 0$ if $n < j$ and $a_j^{(j)} = 1$.

b. Show that $\{f_1, \ldots, f_d\}$ is a basis of S_k. This is called the Miller basis.

c. Show that a cusp form $g \in S_k$ has integral Fourier coefficients if and only if g is a $\mathbb{Z}$-linear combination of the Miller basis elements.