Exercise Sheet 6

1. Let G be a group and let X, Y be sets such that G acts on both X and Y. Consider the *diagonal action* of G on $X \times Y$ given by

$$g \cdot (x, y) = (g \cdot x, g \cdot y).$$

Given $x \in X$ denote by G_x the stabilizer subgroup of G with respect to x. Show that for every subset $S \subseteq X \times Y$ that is G-invariant (i.e. $g \cdot S \subseteq S$ for all $g \in G$) and for every $x \in X$, the set

$$\{y \in Y : (x, y) \in S\} \subseteq Y$$

is G_x-invariant and we have the equality

$$\#(G \backslash S) = \sum_{x \in G \backslash X} \#(G_x \backslash \{y \in Y : (x, y) \in S\}).$$

2. Let $z = x + iy \in \mathbb{H}$ and consider the *theta function*

$$\Theta_z(t) = \sum_{m,n \in \mathbb{Z}} e^{-\pi t \frac{|mz+n|^2}{y}} \quad \text{for } t \in \mathbb{R}^+.$$

Show that Θ_z satisfies the functional equation

$$\Theta_z(t) = \frac{1}{t} \Theta_z\left(\frac{1}{t}\right).$$

3. For $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ define

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \text{g.c.d.}(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

and

$$E^*(z, s) := \pi^{-s} \Gamma(s) 2\zeta(2s) E(z, s) = \pi^{-s} \Gamma(s) \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{y^s}{|mz+n|^{2s}}.$$

a. Check that $E(\gamma z, s) = E(\gamma, s)$ for all $\gamma \in \Gamma$ and show that

$$E^*(z, s) = \int_0^\infty (\Theta_z(t) - 1) t^s \frac{dt}{t}.$$

b. Show that $E^*(z, s)$ has meromorphic continuation to $s \in \mathbb{C}$ with singularities only at $s = 0, 1$, which are simple poles with residues -1 and 1, respectively. Moreover, show that

$$E^*(z, s) = E^*(z, 1 - s).$$

4. Given a fundamental discriminant $D < 0$ let $\mathcal{Q}_D$ denote the set of all positive definite integral quadratic forms $[A, B, C]$ with $B^2 - 4AC = D$, and for every $Q \in \mathcal{Q}_D$ and every integer $n \geq 0$ let $r_Q(n)$ denote the number of primitive representations of n by Q. Recall that Γ acts on $\mathcal{Q}_D$ with finite stabilizers (see Lectures 17 and 18).
Show that for every integer $e \geq 1$ we have

$$r_D(2^e) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{r_Q(2^e)}{|\Gamma_Q|} = \begin{cases} 0 & \text{if } e \geq 2 \text{ and } D \equiv 0 \pmod{4}, \\ 1 + \chi_D(2) & \text{if } e = 1 \text{ or } D \equiv 1 \pmod{4}, \end{cases}$$

where

$$\chi_D(2) = \begin{cases} 0 & \text{if } D \equiv 0 \pmod{4}, \\ 1 & \text{if } D \equiv 1 \pmod{8}, \\ -1 & \text{if } D \equiv 5 \pmod{8}. \end{cases}$$