Solutions Sheet 1

1. **a.** For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with $c \neq 0$ and $z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$ we have

$$\gamma \circ z = \frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}.$$

Taking imaginary parts on both sides we get

$$\mathrm{Im}\,(\gamma \circ z) = \frac{(ad-bc)\cdot \mathrm{Im}(z)}{|cz+d|^2} = \frac{\det(\gamma)\cdot \mathrm{Im}(z)}{|cz+d|^2}.$$

b. The image of the complex upper half-plane $\mathbb{H}$ under the action of γ is $\mathbb{H}$ if $\det(\gamma) > 0$, and it is $\mathbb{H}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ if $\det(\gamma) < 0$.

c. It does not define a group action on $\mathbb{R}$ due to the existence of poles, i.e. if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$ then $\gamma \circ \left(-\frac{d}{c}\right) = \infty \notin \mathbb{R}$. The action must be defined on $\mathbb{R} \cup \{\infty\}$ with the usual conventions concerning the value of $\gamma \circ \infty$ (see Lecture 2).

2. **a.** We have

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = F\left(\left(\frac{a\tau+b}{c\tau+d}\right)\mathbb{Z} + \mathbb{Z}\right) = (c\tau+d)^k F\big((a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z}\big).$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have $(a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z} = \tau\mathbb{Z} + \mathbb{Z}$ (see Lecture 1), thus

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k F(\tau\mathbb{Z}+\mathbb{Z}) = (c\tau+d)^k f(\tau).$$

b. Given $f : \mathbb{H} \to \mathbb{C}$ define $F_f : \mathcal{L} \to \mathbb{C}$ by

$$F_f(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) := \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right), \text{ for } \{\omega_1, \omega_2\} \text{ basis of } \mathbb{R}^2 \text{ with } \frac{\omega_2}{\omega_1} \in \mathbb{H}.$$

The function F_f is well defined since $\omega_1'\mathbb{Z} + \omega_2'\mathbb{Z} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\frac{\omega_1'}{\omega_2'} \in \mathbb{H}$ implies

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some matrix } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

hence

$$(\omega_2')^{-k} f\left(\frac{\omega_1'}{\omega_2'}\right) = (c\omega_1 + d\omega_2)^{-k} f\left(\gamma \circ \frac{\omega_1}{\omega_2}\right) = (c\omega_1 + d\omega_2)^{-k}\left(c\frac{\omega_1}{\omega_2} + d\right)^k f\left(\frac{\omega_1}{\omega_2}\right) = \omega_2^{-1} f\left(\frac{\omega_1}{\omega_2}\right).$$

This proves that F_f is well defined. Now, for $\lambda \in \mathbb{C}^\times$ and $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \in \mathcal{L}$ we have

$$F_f(\lambda L) = F_f(\lambda\omega_1\mathbb{Z} + \lambda\omega_2\mathbb{Z}) = (\lambda\omega_2)^{-k} f\left(\frac{\lambda\omega_1}{\lambda\omega_2}\right) = \lambda^{-k}\omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right) = \lambda^{-k} F_f(L).$$

This proves that F_f satisfies the required transformation property.
We will now show that $f \mapsto F_f$ is the inverse map of $F \mapsto f_F$. Indeed, if $F = F_f$ then

$$f_F(\tau) = F(\tau\mathbb{Z} + \mathbb{Z}) = 1^{-k} f\left(\frac{\tau}{1}\right) = f(\tau),$$

and if $f = f_F$ then we have

$$F_f(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right) = \omega_2^{-k} F\left(\frac{\omega_1}{\omega_2}\mathbb{Z} + \mathbb{Z}\right) = F(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}).$$

3. **a**. We observe that hyperbolas (resp. ellipses, parabolas) are mapped to hyperbolas (resp. ellipses, parabolas) under the linear action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$. Indeed, this can be proven algebraically by noting that a quadratic curve of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

can also be written as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D & E \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F = 0,$$

hence the image of (1) under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the quadratic curve

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A' & B'/2 \\ B'/2 & C' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D' & E' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F = 0,$$

where

$$\begin{pmatrix} A' & B'/2 \\ B'/2 & C' \end{pmatrix} = (\gamma^{-1})^t \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \gamma^{-1}, \text{ and } \begin{pmatrix} D' & E' \end{pmatrix} = \begin{pmatrix} D & E \end{pmatrix} \gamma^{-1}.$$

This implies

$$\det \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} = \det \begin{pmatrix} A' & B'/2 \\ B'/2 & C' \end{pmatrix},$$

hence $B^2 - 4AC = (B')^2 - 4(A')(C')$. By the distinction of conics according to the sign of the *discriminant* $B^2 - 4AC$ we conclude that hyperbolas (resp. ellipses, parabolas) are mapped to hyperbolas (resp. ellipses, parabolas).

Now, let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ be a matrix different from $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $M \in \mathrm{SL}_2(\mathbb{R})$ then the invariant sets of $M\gamma M^{-1}$ are all of the form $M \circ Y$ where Y is an invariant set of γ. Thus, it is enough to prove the result for $M\gamma M^{-1}$. In what follows we use some results presented in Lecture 2 of the course.

If γ is elliptic, then it is conjugated to a matrix of the form $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ with $\theta \in \mathbb{R}$. This acts on $\mathbb{R}^2$ as a rotation around the origin, hence all circles (which are special cases of ellipses) centered at $(0, 0)$ are invariant.

If γ is hyperbolic, then it is conjugated to matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R}, \lambda \neq 0, 1, -1$. This acts on $\mathbb{R}^2$ leaving invariant each hyperbola of equation $xy = t$, with $t \in \mathbb{R}, t \neq 0$ a parameter.

b. If γ is parabolic, then it is conjugated to matrix of the form $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ with $\lambda \in \mathbb{R}, \lambda \neq 0$. This acts on $\mathbb{R}^2$ leaving invariant each line of equation $y = t$, with $t \in \mathbb{R}$ a parameter.

4. Put $\gamma = \begin{pmatrix} 4 & -9 \\ -11 & 25 \end{pmatrix}$. We follow the algorithm presented in Lecture 3. First we write

$$25 = (-11) \cdot (-2) + 3$$

and compute

$$\gamma T^2 S = \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix} S = \begin{pmatrix} -1 & -4 \\ 3 & 11 \end{pmatrix}.$$

Now we write $11 = 3 \cdot 3 + 2$ and compute

$$\gamma T^2 S T^{-3} S = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} S = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}.$$

We write $-3 = 2 \cdot (-2) + 1$ and compute

$$\gamma T^2 S T^{-3} S T^2 S = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} S = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}.$$

Finally, we write $-2 = 1 \cdot (-2)$ and compute

$$\gamma T^2 S T^{-3} S T^2 S T^2 S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = S^2 T^{-1}.$$

We get $\gamma = S^2 T^{-1} S T^{-2} S T^{-2} S T^3 S T^{-2}$.

5. **a.** The group Γ_θ contains the subgroup

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\},$$

which is of index 6 in $\mathrm{SL}_2(\mathbb{Z})$ since it is the kernel of the reduction mod 2 map to $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, and the group $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is the image of the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

We have

$$\Gamma_\theta = \Gamma(2) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma(2),$$

hence $[\Gamma_\theta : \Gamma(2)] = 2$. It follows that Γ_θ has index 3 in $\mathrm{SL}_2(\mathbb{Z})$.

b. A set of representatives for $\Gamma_\theta \backslash \mathrm{SL}_2(\mathbb{Z})$ is given by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Hence, letting $F = \{\tau \in \mathbb{H} : |\tau| \geq 1, |\mathrm{Re}(\tau)| \leq \frac{1}{2}\}$ be the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ (see Lecture 3), we get that a fundamental domain for Γ_θ is

$$F' = F \cup T(F) \cup TS(F).$$

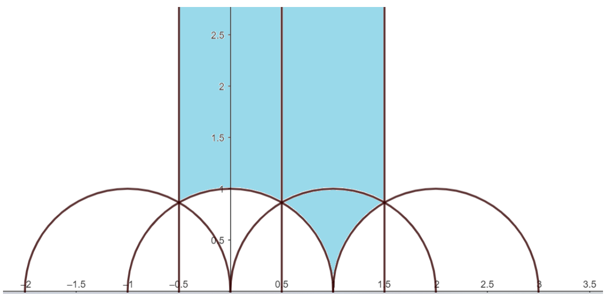

FIGURE 1. Fundamental domain F'

We note that $F' = F_1 \cup F_2$ where

$$F_1 := \left\{ \tau \in \mathbb{H} : |\tau| \geq 1, -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 1 \right\}, F_2 := \left\{ \tau \in \mathbb{H} : |\tau - 2| \geq 1, 1 \leq \mathrm{Re}(\tau) \leq \frac{3}{2} \right\}.$$

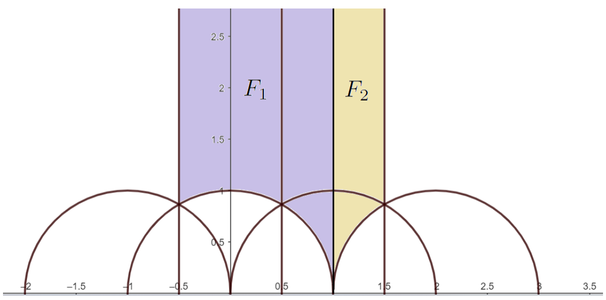

FIGURE 2. Decomposition of F' into F_1 and F_2

We have

$$F_\theta = F_1 \cup T^{-2} F_2,$$

hence F_θ is a fundamental domain for Γ_θ.

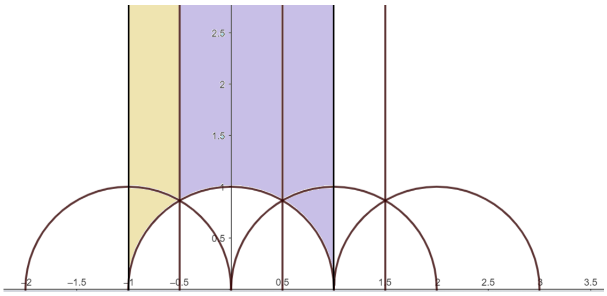

FIGURE 3. Fundamental domain F_θ

c. We will use that F_θ is a fundamental domain for Γ_θ, hence for every point z in the interior of F_θ we have

$$\gamma_0 \in \Gamma_\theta, \gamma_0 \circ z \in F_\theta \Rightarrow \gamma_0 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\gamma \in \Gamma_\theta$ and choose z a point in the interior of F_θ (e.g. $z = 2i$). Put $z_0 = \gamma \circ z$. Choose $n_1 \in \mathbb{Z}$ so that $z_1 := T^{2n_1} \circ z_0$ lies in $R := \{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq 1\}$. If $|z_1| < 1$ then apply S to z_1 and choose $n_2 \in \mathbb{Z}$ so that $z_2 := T^{2n_2} S \circ z_1 \in R$. Repeat this process as many times as necessary to end up with a point of the form

$$z_k = (T^{2n_k} S \cdots T^{2n_3} S T^{2n_2} S T^{2n_1}) \circ z_0$$

in F_θ. Note that this process must end after finitely many steps since

$$\mathrm{Im}(z_{t+1}) = \frac{\mathrm{Im}(z_t)}{|z_t|^2}, \text{ for all } t \in \{1, \ldots, k-1\} \text{ if } k \geq 1,$$

hence $\mathrm{Im}(z_0) = \mathrm{Im}(z_1) < \mathrm{Im}(z_2) < \ldots$, but

$$\mathrm{Im}(M \circ z_0) = \frac{\mathrm{Im}(z_0)}{|cz_0 + d|^2}, \text{ if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta,$$

and $|cz_0 + d| < 1$ gives finitely many possibilities for the integers c and d. Since z_k is in F_θ and it is Γ_θ-equivalent to z_0 we conclude

$$T^{2n_k} S \cdots T^{2n_3} S T^{2n_2} S T^{2n_1} \gamma = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, this implies that γ is in the group generated by T^2 and S.