Solutions Sheet 2

1. **a.** For $z \in \mathbb{C} \setminus \{0\}$ have

$$\frac{\sin(z)}{z} = \frac{e^{iz} - e^{-iz}}{2iz} = \lim_{n\to\infty} f_n(z)$$

where

$$f_n(z) = \frac{\left(1 + \frac{iz}{n}\right)^n - \left(1 - \frac{iz}{n}\right)^n}{2iz}.$$

If $n = 2m + 1$ with $m \in \mathbb{Z}^+$ then $f_n(x)$ is a polynomial of degree $n - 1 = 2m$ with constant term 1 and whose roots are of the form

$$z = -in\left(\frac{\zeta - 1}{\zeta + 1}\right) \text{ where } \zeta^n = 1, \zeta \neq 1.$$

If follows that

$$f_n(z) = \prod_{\zeta^n=1,\zeta\neq 1} \left(1 - \frac{iz}{n}\left(\frac{\zeta + 1}{\zeta - 1}\right)\right).$$

Writing $\zeta = e^{2\pi ik/n}$ with $k \in \{1, \ldots, n - 1 = 2m\}$ we get

$$\begin{aligned} f_n(z) &= \prod_{k=1}^{m} \left(1 - \frac{iz}{n}\left(\frac{e^{2\pi ik/n} + 1}{e^{2\pi ik/n} - 1}\right)\right)\left(1 - \frac{iz}{n}\left(\frac{e^{-2\pi ik/n} + 1}{e^{-2\pi ik/n} - 1}\right)\right) \\ &= \prod_{k=1}^{m} \left(1 - \frac{z^2}{n^2}\left(\frac{1 + \cos(2\pi k/n)}{1 - \cos(2\pi k/n)}\right)\right). \end{aligned}$$

This implies

$$\frac{\sin(z)}{z} = \lim_{\substack{n\to\infty \\ n=2m+1}} \prod_{k=1}^{m} \left(1 - \frac{z^2}{n^2}\left(\frac{1 + \cos(2\pi k/n)}{1 - \cos(2\pi k/n)}\right)\right) \text{ for all } z \in \mathbb{C} \setminus \{0\}. \tag{1}$$

Now, in order to get the desired infinite product, let us introduce the following auxiliary functions

$$\begin{aligned} g_{n,r}(x) &:= \prod_{k=1}^{r} \left(1 + \frac{x^2}{n^2}\left(\frac{1 + \cos(2\pi k/n)}{1 - \cos(2\pi k/n)}\right)\right), \\ g_n(x) &:= \prod_{k=1}^{m} \left(1 + \frac{x^2}{n^2}\left(\frac{1 + \cos(2\pi k/n)}{1 - \cos(2\pi k/n)}\right)\right) = g_{n,m}(x), \end{aligned}$$

for $x \in \mathbb{R}$ where $1 \leq r \leq m$ are integers and $n = 2m + 1$. Note that by (1) we have

$$\frac{\sin(ix)}{ix} = \lim_{n\to\infty} g_n(x) \text{ for all } x \in \mathbb{R} \setminus \{0\}. \tag{2}$$

Now, we compute

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{1 + \cos(2\pi k/n)}{1 - \cos(2\pi k/n)}\right) &= \lim_{n\to\infty} \frac{1}{n^2} \frac{(1 + \cos(2\pi k/n))^2}{\sin(2\pi k/n)^2} \\ &= (2\pi k)^{-2} \lim_{n\to\infty} \left(\frac{2\pi k}{n}\right)^2 \frac{(1 + \cos(2\pi k/n))^2}{\sin(2\pi k/n)^2} \\ &= \frac{1}{\pi^2 k^2}. \end{aligned}$$

Hence, since $g_{n,r}(x) \leq g_n(x)$, taking $n \to \infty$ with r fixed and using (2) we get

$$\prod_{k=1}^{r} \left(1 + \frac{x^2}{k^2\pi^2}\right) \leq \frac{\sin(ix)}{ix},$$

and taking $r \to \infty$ gives

$$\prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right) \leq \frac{\sin(ix)}{ix}.$$

Now, one can check that the function $t \mapsto t^2 \left(\frac{1+\cos(t)}{1-\cos(t)}\right)$ is decreasing for $t \in]0, \pi[$ (e.g., using derivatives). This implies that

$$\frac{1}{n^2}\left(\frac{1+\cos(2\pi k/n)}{1-\cos(2\pi k/n)}\right) \leq \frac{1}{\pi^2 k^2} \quad \text{for } 1 \leq k \leq m,$$

hence

$$g_n(x) \leq \prod_{k=1}^{m} \left(1 + \frac{x^2}{k^2\pi^2}\right),$$

and taking $n \to \infty$, using (2) again, gives the inequality

$$\frac{\sin(ix)}{ix} \leq \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right).$$

We conclude that

$$\frac{\sin(ix)}{ix} = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right) \quad \text{for all } x \in \mathbb{R} \setminus \{0\}. \tag{3}$$

Since the functions

$$\frac{\sin(z)}{z} \quad \text{and} \quad \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

are analytic for $z \in \mathbb{C} \setminus \{0\}$ (see *Complex Analysis* by S. Lang (Springer 1999), chapter XIII), and they coincide for $z \in i(\mathbb{R} \setminus \{0\})$, we conclude that

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \quad \text{for } z \in \mathbb{C} \setminus \{0\}. \tag{4}$$

Added comment: Instead of using the analyticity of infinite products, one can extend the above computations and prove that (4) holds directly. Indeed, we already proved (3). Extending the definitions of $g_{n,r}(x)$ and $g_n(x)$ to any $x = z \in \mathbb{C} \setminus \{0\}$ we have

$$\begin{aligned} |g_n(z) - g_{n,r}(z)| &= |g_{n,r}(z)| \cdot \left| \prod_{k=r+1}^{m} \left(1 + \frac{z^2}{n^2}\left(\frac{1+\cos(2\pi k/n)}{1-\cos(2\pi k/n)}\right)\right) - 1 \right| \\ &\leq g_{n,r}(|z|) \cdot \left(\prod_{k=r+1}^{m} \left(1 + \frac{|z|^2}{n^2}\left(\frac{1+\cos(2\pi k/n)}{1-\cos(2\pi k/n)}\right)\right) - 1 \right) \\ &= g_n(|z|) - g_{n,r}(|z|). \end{aligned}$$

Taking $n \to \infty$ with r fixed and using (2) we get

$$\left| \frac{\sin(iz)}{iz} - \prod_{k=1}^{r} \left(1 + \frac{z^2}{k^2\pi^2}\right) \right| \leq \frac{\sin(i|z|)}{i|z|} - \prod_{k=1}^{r} \left(1 + \frac{|z|^2}{k^2\pi^2}\right).$$

As $r \to \infty$ the right hand side converges to 0 by 3, hence

$$\frac{\sin(iz)}{iz} = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right).$$

Replacing z by $-iz$ gives (4).

b. Taking logarithmic derivatives on both sides of (4) we get

$$\begin{aligned}
\cot(z) - \frac{1}{z} &= \sum_{n=1}^{\infty} \frac{-2z}{n^2\pi^2}\left(1 - \frac{z^2}{n^2\pi^2}\right)^{-1} \\
&= -2z \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n\pi + z} - \frac{1}{n\pi - z}\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n\pi + z} - \frac{1}{\pi n}\right) - \sum_{n=1}^{\infty} \left(\frac{1}{n\pi - z} - \frac{1}{\pi n}\right) \\
&= \sum_{n\geq 1} \left(\frac{1}{n\pi + z} - \frac{1}{\pi n}\right) + \sum_{n\leq -1} \left(\frac{1}{n\pi + z} - \frac{1}{\pi n}\right) \\
&= \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{n\pi + z} - \frac{1}{\pi n}\right).
\end{aligned}$$

Replacing z by πz we get

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z+n} - \frac{1}{n}\right). \tag{5}$$

c. From (5) we have

$$\begin{aligned}
\pi z\cot(\pi z) &= 1 + z \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z+n} - \frac{1}{n}\right) \\
&= 1 + z \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right) \\
&= 1 + 2z^2 \sum_{n=1}^{\infty} \left(\frac{1}{z^2 - n^2}\right) \\
&= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1}{1-(z/n)^2}\right) \\
&= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{z}{n}\right)^{2k} \\
&= 1 - 2z^2 \sum_{k=0}^{\infty} \zeta(2k+2) z^{2k} \\
&= 1 - 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k}.
\end{aligned}$$

d. We have

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\,\frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} = i + \frac{2i}{e^{2\pi iz} - 1}. \tag{6}$$

For $z \in \mathbb{H}$ we get

$$\pi z\cot(\pi z) = \pi iz + \frac{2\pi iz}{e^{2\pi iz} - 1} = \pi iz + \sum_{k=0}^{\infty} \frac{B_k}{k!}(2\pi iz)^k = B_0 + \pi i(1 + 2B_1)z + \sum_{k=2}^{\infty} \frac{(2\pi i)^k B_k}{k!} z^k.$$

Hence

$$1 - 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k} = B_0 + \pi i(1 + 2B_1)z + \sum_{k=2}^{\infty} \frac{(2\pi i)^k B_k}{k!} z^k,$$

which implies

$$-2\zeta(k) = \frac{(2\pi i)^k B_k}{k!} \text{ for } k \in \mathbb{Z}^+ \text{ even.}$$

2. **a**. We have

$$\frac{1}{(m-1+n\tau)(m+n\tau)} = \frac{1}{(m-1+n\tau)} - \frac{1}{(m+n\tau)}.$$

By recognizing telescopic series, we get

$$\sum_{m\in\mathbb{Z}}{}' \frac{1}{(m-1+n\tau)(m+n\tau)} = \begin{cases} 0 & \text{if } n \neq 0, \\ 2 & \text{if } n = 0. \end{cases} \tag{7}$$

This implies that $H_1(\tau) = 2$. Now, in order to compute $H_2(\tau)$ we assume $m \neq 0, 1$ and write

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \frac{1}{(m-1+n\tau)(m+n\tau)} \\ = \;& \tau^{-1}\left(\sum_{n\neq 0}\left(\frac{1}{((m-1)\tau^{-1}+n)} - \frac{1}{n}\right) - \sum_{n\neq 0}\left(\frac{1}{(m\tau^{-1}+n)} - \frac{1}{n}\right)\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right). \end{aligned}$$

Using (5) we get

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \frac{1}{(m-1+n\tau)(m+n\tau)} \\ = \;& \tau^{-1}\left(\pi\cot\left(\frac{\pi(m-1)}{\tau}\right) - \frac{\tau}{m-1} - \pi\cot\left(\frac{\pi m}{\tau}\right) + \frac{\tau}{m}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ = \;& \tau^{-1}\left(\pi\cot\left(\frac{\pi(m-1)}{\tau}\right) - \pi\cot\left(\frac{\pi m}{\tau}\right)\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\sum_{m\in\{0,1\}}\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(m-1+n\tau)(m+n\tau)} \\ = \;& \tau^{-1}\left(\sum_{n\neq 0}\left(\frac{1}{(-\tau^{-1}+n)} - \frac{1}{n}\right) - \sum_{n\neq 0}\left(\frac{1}{(\tau^{-1}+n)} - \frac{1}{n}\right)\right) \\ = \;& \tau^{-1}\left(\pi\cot\left(\frac{\pi(-1)}{\tau}\right) - \pi\cot\left(\frac{\pi}{\tau}\right) + 2\tau\right) \\ = \;& \tau^{-1}\left(\pi\cot\left(\frac{\pi(-1)}{\tau}\right) - \pi\cot\left(\frac{\pi}{\tau}\right)\right) + 2. \end{aligned}$$

Hence, by recognizing telescopic series again, we get

$$H_2(\tau) = 2 + \tau^{-1}\pi\left(\lim_{N\to-\infty}\cot\left(\frac{\pi N}{\tau}\right) - \lim_{M\to\infty}\cot\left(\frac{\pi M}{\tau}\right)\right) = 2 - \frac{2\pi i}{\tau}$$

where, in the computation of the limits, we used (6).
Finally, the identity $F_1 - H_1 = F_2 - H_2$ follows from the fact that

$$\frac{1}{(m+n\tau)^2} - \frac{1}{(m-1+n\tau)(m+n\tau)} = -\frac{1}{(m+n\tau)^2(m-1+n\tau)}$$

is absolutely summable over (n, m), hence the corresponding double series can be re-arranged at our convenience.

b. It follows directly from **a.** that $F_1(\tau) - F_2(\tau) = H_1(\tau) - H_2(\tau) = \frac{2\pi i}{\tau}$. We now compute

$$
\begin{aligned}
F_1\left(-\frac{1}{\tau}\right) &= \tau^2 \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{(\tau m - n)^2}, \\
&= \tau^2 \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{(n + m\tau)^2}, \\
&= \tau^2 F_2(\tau) \\
&= \tau^2 \left(F_1(\tau) - \frac{2\pi i}{\tau}\right) \\
&= \tau^2 F_1(\tau) - 2\pi i \tau.
\end{aligned}
$$

c. We follow the computation of the Fourier expansion of Eisenstein series given in Lecture 5. From the first equality in (6) with $z = \tau \in \mathbb{H}$ we have

$$\cot(\pi\tau) = -i - 2i \sum_{k=1}^{\infty} e^{2\pi i k \tau}.$$

Taking derivatives on both sides and using (5) we get

$$\sum_{m\in\mathbb{Z}} \frac{1}{(\tau + m)^2} = 4\pi \sum_{k=1}^{\infty} k e^{2\pi i k \tau}.$$

This implies

$$
\begin{aligned}
F_1(\tau) &= 2\sum_{m=1}^{\infty} \frac{1}{m^2} + 2\sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(n\tau + m)^2} \\
&= 2\zeta(2) + 8\pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k e^{2\pi i k n \tau} \\
&= \frac{\pi^2}{3} + 8\pi \sum_{N=1}^{\infty} \sigma_1(N) e^{2\pi i N \tau} \\
&= G_2(\tau).
\end{aligned}
$$

3. E_4^2, E_8 are non-zero vectors in M_8 and this space has dimension 1, hence $E_4^2 = \lambda E_8$ for some $\lambda \in \mathbb{C}^\times$. Since E_4 and E_8 are normalized to have 0-th Fourier coefficient equal to 1, we have $\lambda = 1$. Similar arguments, using that the spaces M_{10} and M_{14} are one dimensional, imply that $E_4 E_6 = E_{10}$ and $E_6 E_8 = E_{14}$.
Using

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

(see Lecture 5) we get from $E_4^2 = E_8$ the identity

$$480\sigma_7(n) = 480\sigma_3(n) + (240)^2 \sum_{k=1}^{n-1} \sigma_3(n-k)\sigma_3(k) \text{ for all integers } n \geq 2,$$

which simplifies to

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(n-k)\sigma_3(k) \text{ for all integers } n \geq 2.$$

In a similar way, identities $E_4E_6 = E_{10}$ and $E_6E_8 = E_{14}$ imply

$$\begin{aligned} 11\sigma_9(n) &= 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{k=1}^{n-1} \sigma_3(n-k)\sigma_5(k), \\ \sigma_{13}(n) &= 21\sigma_5(n) - 20\sigma_7(n) + 10080 \sum_{k=1}^{n-1} \sigma_5(n-k)\sigma_7(k), \end{aligned}$$

for all integers $n \geq 2$.

4. **a**. We have $E_2 = \frac{3}{\pi^2} G_2$ hence

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau.$$

Given f in M_k we have

$$f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau),$$

hence

$$E_2\left(-\frac{1}{\tau}\right) f\left(-\frac{1}{\tau}\right) = \tau^{k+2} E_2(\tau) f(\tau) - \frac{6i}{\pi}\tau^{k+1} f(\tau),$$

and

$$f'\left(-\frac{1}{\tau}\right)\frac{1}{\tau^2} = k\tau^{k-1} f(\tau) + \tau^k f'(\tau).$$

This implies

$$\begin{aligned} g\left(-\frac{1}{\tau}\right) &= \frac{1}{2\pi i}\left(k\tau^{k+1} f(\tau) + \tau^{k+2} f'(\tau)\right) - \frac{k}{12}\left(\tau^{k+2} E_2(\tau) f(\tau) - \frac{6i}{\pi}\tau^{k+1} f(\tau)\right) \\ &= \frac{1}{2\pi i}\tau^{k+2} f'(\tau) - \frac{k}{12}\tau^{k+2} E_2(\tau) f(\tau) \\ &= \tau^{k+2} g(\tau). \end{aligned}$$

Since we also have $g(\tau+1) = g(\tau)$ (since f, f' and E_2 are invariant under $\tau \mapsto \tau + 1$), and $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we conclude that g transforms like a modular form of weight $k+2$ for $\mathrm{SL}_2(\mathbb{Z})$. By construction, g is clearly holomorphic in $\mathbb{H}$. Finally, since f' has no constant term in its Fourier expansion, we get

$$\lim_{\tau \to i\infty} g(\tau) = -\frac{k}{12} a_0$$

where a_0 is the 0-th Fourier coefficient of f. This proves that g is holomorphic at $i\infty$, hence $g \in M_{k+2}$, and also that g is cuspidal if and only if f is cuspidal.

b. When $f = E_4$ we have $g \in M_6$ with 0-th Fourier coefficient $-\frac{1}{3}$. Since M_6 is one dimensional, we have $g = -\frac{1}{3}E_6$. Similarly, one prove that $g = -\frac{1}{2}E_8$ when $f = E_6$ and $g = 0$ when $f = \Delta$. This implies

$$\begin{aligned} 21\sigma_5(n) &= (30n - 10)\sigma_3(n) - \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(n-m)\sigma_3(m), \\ 20\sigma_7(n) &= (42n - 21)\sigma_5(n) - \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(n-m)\sigma_5(m), \\ (n-1)\tau(n) &= -24 \sum_{m=1}^{n-1} \sigma_1(n-m)\tau(m), \end{aligned}$$

for all integers $n \geq 2$.