Solutions Sheet 3

1. **a.** Write $p(X,Y)=\sum_{a,b\geq 0} c_{a,b}X^aY^b$. There exists $N\in\mathbb{Z}^+$ such that

$$p(X,Y)=\sum_{k=0}^{N} p_k(X,Y) \text{ where } p_k(X,Y):=\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b}X^aY^b.$$

Then $f_k := p_k(E_4,E_6)$ is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. We first show that $f_k=0$ for all k. The result is clear if $N=0$, so we can assume $N\geq 1$. From $p(E_4,E_6)=0$ we have

$$\sum_{k=0}^{N} f_k(\tau)=0.$$

Using that $f_k\left(-\frac{1}{\tau}\right)=\tau^k f_k(\tau)$ we get

$$\sum_{k=0}^{N} \tau^k f_k(\tau)=0.$$

Now, replacing τ by $\tau+t$ for $t\in\{0,\dots,N\}$, we get

$$M(\tau)\cdot\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_N\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\ 0\end{pmatrix}$$

where

$$M(\tau):=\begin{pmatrix} 1 & \tau & \cdots & \tau^N\\ 1 & \tau+1 & \cdots & (\tau+1)^N\\ \vdots & \vdots & \cdots & \vdots\\ 1 & \tau+N & \cdots & (\tau+N)^N\end{pmatrix}.$$

The matrix $M(\tau)$ is of Vandermonde type. Since $1,\tau,\tau+1,\dots,\tau+N$ are all different, we have $\det(M(\tau))\neq 0$. This implies $f_k=0$ for all k.
Now, since $p_k(E_4,E_6)=f_k=0$ we have

$$\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b}(E_4)^a(E_6)^b=0.$$

Let $U\subseteq\mathbb{H}$ be a small open disk containing no zeroes of E_4 and no zeroes of E_6 (actually, using the valence formula from Lecture 6, one can prove that the only zeroes of E_4 and of E_6 are the points which are $\mathrm{SL}_2(\mathbb{Z})$-equivalent to $\frac{-1+i\sqrt{3}}{2}$ and to i, respectively). Then, we can consider the holomorphic functions $f:=E_4^{1/2}$ and $g:=E_6^{1/3}$ on U satisfying

$$\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b}f^{2a}g^{3b}=0.$$

Dividing by $g^{k/2}$ gives

$$P_k\left(\frac{f}{g}\right)=0, \text{ where } P_k(X):=\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b}X^{2a}.$$

If the polynomial $P_k(X)$ is non-zero, then it has finitely many roots, and the above equality implies $\frac{f}{g}=\lambda$ in U for some root λ of $P_k(X)$. This implies that $E_4^3=\lambda^6E_6^2$ in U, hence also in $\mathbb{H}$ by the identity principle. But E_4^3 and E_6^2 are linearly independent in M_{12} (this can be seen easily comparing the Fourier expansions of these two forms, or by evaluating at $\frac{-1+i\sqrt{3}}{2}$ and i). We conclude that $P_k(X)=0$, which implies $p_k(X)=0$.

b. By part **a**. we know that all elements in the set

$$\{E_4^a E_6^b : a, b \in \mathbb{Z}_0^+, 4a + 6b = k\} \tag{1}$$

are linearly independent. The number of such products of Eisenstein series is

$$\#\{(a,b) \in (\mathbb{Z}_0^+)^2 : 4a + 6b = k\} = \#\{n \in \{0, 1, \ldots, k\} : n \equiv 0 \text{ (mod 4) and } n \equiv k \text{ (mod 6)}\}.$$

If $k \equiv 0, 2, 4, 6, 8, 10 \pmod{12}$, let $k' := 0, 8, 4, 0, 8, 4$ respectively. Then

$$n \equiv 0 \,(\text{mod } 4) \text{ and } n \equiv k \,(\text{mod } 6) \Leftrightarrow n \equiv k' \pmod{12}.$$

If r is the rest of dividing k by 12, so $r \equiv k \pmod{12}$, we get

$$\begin{aligned}
&\#\{(a,b) \in (\mathbb{Z}_0^+)^2 : 4a + 6b = k\} \\
= \;& \#\{n \in \{0, 1, \ldots, k\} : n \equiv 0 \,(\text{mod } 4) \text{ and } n \equiv k \,(\text{mod } 6)\} \\
= \;& \#\{n \in \{0, 1, \ldots, k\} : n \equiv k' \,(\text{mod } 12\} \\
= \;& \left\lfloor \frac{k}{12} \right\rfloor + \#\{n \in \{0, 1, \ldots, r\} : n \equiv k' \,(\text{mod } 12)\} \\
= \;& \left\lfloor \frac{k}{12} \right\rfloor + \begin{cases} 1 & \text{if } k \equiv 0, 4, 6, 8, 10 \pmod{12}, \\ 0 & \text{if } k \equiv 2 \pmod{12}. \end{cases}
\end{aligned}$$

This equals the dimension of M_k, hence the set (1) is a basis of M_k.

2. **a**. The logarithmic derivative of F is

$$\begin{aligned}
\log(F(\tau))' &= 2\pi i + 24 \sum_{n=1}^{\infty} (-2\pi i n) \frac{q^n}{1 - q^n} \\
&= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} n \sum_{k=1}^{\infty} q^{kn} \right) \\
&= 2\pi i \left(1 - 24 \sum_{N=1}^{\infty} \sigma_1(N) q^N \right) \\
&= 2\pi i E_2(\tau).
\end{aligned}$$

b. Since

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau$$

we have

$$\begin{aligned}
\log\left(F\left(\frac{-1}{\tau}\right)\tau^{-12}\right)' &= \left(\log\left(F\left(\frac{-1}{\tau}\right)\right) - 12\log(\tau)\right)' \\
&= 2\pi i E_2\left(\frac{-1}{\tau}\right)\frac{1}{\tau^2} - 12\frac{1}{\tau} \\
&= 2\pi i \left(\tau^2 E_2(\tau) - \frac{6i}{\pi}\tau\right)\frac{1}{\tau^2} - 12\frac{1}{\tau} \\
&= 2\pi i E_2(\tau) \\
&= \log(F(\tau))'.
\end{aligned}$$

This implies that $F\left(\frac{-1}{\tau}\right)\tau^{-12} = \lambda F(\tau)$ for some $\lambda \in \mathbb{C}$. Evaluating at $\tau = i$ we see that $\lambda = 1$. Since $F(\tau)$ is also invariant under $\tau \mapsto \tau + 1$, because it is defined in terms of $q = e^{2\pi i \tau}$, we conclude that $F \in S_{12}$.

c. We have $S_{12} = \mathbb{C}\Delta$ and both F and Δ have first Fourier coefficient equal to 1, hence $F = \Delta$. This completes the proof of the identity

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \text{ where } q := e^{2\pi i \tau}. \tag{2}$$

3. **a.** We have

$$\begin{aligned} E_6(\tau) &= 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \\ E_{12}(\tau) &= 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n, \end{aligned}$$

hence $E_{12} - E_6^2$ is in S_{12} and has first Fourier coefficient

$$c = \frac{65520}{691} + 2 \cdot 504 = \frac{2^6\, 3^5\, 7^2}{691}.$$

Since $S_{12} = \mathbb{C}\Delta$, we conclude that $E_{12} - E_6^2 = c\Delta$.

b. Let $a(n)$ denote the n-th Fourier coefficient of E_6^2. Since E_6 has Fourier coefficients in $\mathbb{Z}$, we have that $a(n) \in \mathbb{Z}$ for all n. From **a**. we have

$$65520\, \sigma_{11}(n) - 691\, a(n) = (2^6\, 3^5\, 7^2)\tau(n) \text{ for all } n \in \mathbb{Z}^+.$$

Since $65520 \equiv 2^6\, 3^5\, 7^2 \not\equiv 0 \pmod{691}$, and 691 is prime, we conclude

$$\sigma_{11}(n) \equiv \tau(n) \pmod{691} \text{ for all } n \in \mathbb{Z}^+.$$

4. **a.** By definition (see Lecture 7)

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}.$$

We mentioned in **1.a** that $E_4(\rho) = 0$ where $\rho := -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, hence $j(\rho) = 0$. By the definition of Δ we also have

$$j(\tau) = 1728\frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2},$$

hence evaluating at $\tau = i$, using that $E_6(i) = 0$ and $E_4(i) \neq 0$, we get $j(i) = 1728$. Finally, the infinite product expansion (2) and the Fourier expansion

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

imply that $j(\tau)$ has Fourier expansion

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty}c(n)q^n \text{ with } c(n) \in \mathbb{Z}_0^+.$$

Hence

$$\overline{j(\tau)} = \frac{1}{\overline{q}} + \sum_{n=0}^{\infty}c(n)\overline{q}^n.$$

Since $\overline{q} = e^{2\pi i(-\overline{\tau})}$ we get $\overline{j(\tau)} = j(-\overline{\tau})$.

b. For $t > 0$ we have

$$j(it) = e^{2\pi t} + \sum_{n=0}^{\infty}c(n)e^{-2\pi nt} > 0.$$

Since j is continuous, injective when restricted to C, $j(i) = 1728$ and

$$\lim_{t\to\infty} j(it) = \infty,$$

we conclude that $j(C) =]1728, \infty[$. Now, for $\tau \in B$ we have, using $j(\tau) = \overline{j(-\overline{\tau})}$, the equalities

$$j(\tau) = j\left(-\frac{1}{\tau}\right) = j(-\overline{\tau}) = \overline{j(\tau)}.$$

This proves that $j(B) \subseteq \mathbb{R}$. As before, since j is continuous, injective when restricted to B, $j(i) = 1728$ and $j(\rho) = 0$, we conclude that $j(B) =]0, 1728[$. Finally, for $t > 0$ we have

$$j\left(-\frac{1}{2} + it\right) = j\left(-\frac{1}{2} + it + 1\right) = j\left(\frac{1}{2} + it\right).$$

Using $j(\tau) = \overline{j(-\overline{\tau})}$ we get

$$j\left(-\frac{1}{2} + it\right) = \overline{j\left(-\frac{1}{2} + it\right)}.$$

This implies that $j(A) \subseteq \mathbb{R}$. Moreover, we have

$$j\left(\frac{1}{2} + it\right) = -e^{2\pi t} + \sum_{n=0}^{\infty} c(n)(-1)^n e^{-2\pi n t}$$

which implies

$$\lim_{t\to\infty} j\left(\frac{1}{2} + it\right) = -\infty.$$

Using similar arguments as before, we conclude that $j(A) =]-\infty, 0[$.

c. Recall that j induces a bijection between $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and $\mathbb{C}$. Since $A \cup B \cup C \cup \{\rho, i\} \cup \mathcal{F}_1 \cup \mathcal{F}_2$ has exactly one representative of each $\mathrm{SL}_2(\mathbb{Z})$ orbit in $\mathbb{H}$, and

$$j(A \cup B \cup C \cup \{\rho, i\}) = \mathbb{R},$$

we have that $j(\mathcal{F}_1) = \mathbb{H}$ and $j(\mathcal{F}_2) = \mathbb{H}^-$, or $j(\mathcal{F}_1) = \mathbb{H}^-$ and $j(\mathcal{F}_2) = \mathbb{H}$. Since j is holomorphic, it preserves orientations. Hence $j(\mathcal{F}_1) = \mathbb{H}$ and $j(\mathcal{F}_2) = \mathbb{H}^-$.