# Solutions Sheet 4

**1.** **a.** We have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dxdy &= \int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy\\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2\\ &= \left(2\int_{0}^{\infty} e^{-x^2}dx\right)^2\\ &= \left(\int_{0}^{\infty} e^{-t}t^{-\frac{1}{2}}dt\right)^2\\ &= \Gamma\left(\frac{1}{2}\right)^2.\end{aligned}$$

On the other hand, using polar coordinates we have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dxdy &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy\\ &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}rdrd\theta\\ &= 2\pi\frac{-e^{-r^2}}{2}\Bigg|_{r=0}^{r=\infty}\\ &= \pi.\end{aligned}$$

This proves that $\Gamma\left(\frac{1}{2}\right)^2=\pi$. Since $\Gamma\left(\frac{1}{2}\right)>0$ we conclude $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

**b.** For $s_1, s_2 \in H$ we have

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_0^\infty e^{-x}x^{s_1-1}dx\int_0^\infty e^{-y}y^{s_2-1}dy\\ &= \int_{]0,\infty[^2} e^{-x-y}x^{s_1-1}y^{s_2-1}dxdy.\end{aligned}$$

Using the change of variables $]0,\infty[\times]0,1[\to]0,\infty[^2, (u,v)\mapsto(uv,u(1-v))$, we get

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) &= \int_0^\infty\int_0^1 ue^{-u}u^{s_1+s_2-2}v^{s_1-1}(1-v)^{s_2-1}dudv\\ &= \Gamma(s_1+s_2)\int_0^1 v^{s_1-1}(1-v)^{s_2-1}dv.\end{aligned}$$

We conclude

$$B(s_1,s_2)=\int_0^1 v^{s_1-1}(1-v)^{s_2-1}dv.$$

**c**. Let $s \in H$. Using the change of variables $v = \frac{1+x}{2}$ and $t = x^2$ we have

$$\begin{aligned} B(s,s) &= \int_0^1 v^{s-1}(1-v)^{s-1}dv \\ &= \frac{1}{2}\int_{-1}^1 \left(\frac{1+x}{2}\right)^{s-1}\left(\frac{1-x}{2}\right)^{s-1} dx \\ &= 2^{1-2s}\int_{-1}^1 (1-x^2)^{s-1}dx \\ &= 2^{2-2s}\int_0^1 (1-x^2)^{s-1}dx \\ &= 2^{1-2s}\int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}dt \\ &= 2^{1-2s}B\left(\frac{1}{2},s\right). \end{aligned}$$

We conclude

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = 2^{1-2s}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}.$$

Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we get

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s). \tag{1}$$

**2**. **a**. By the duplication formula (1) with $s$ replaced by $\frac{s}{2}$ we have

$$\begin{aligned} \Lambda_k(s) &= (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-k+1) \\ &= \frac{1}{2}\pi^{-s-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s)\zeta(s-k+1). \end{aligned}$$

Put $\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Then

$$\Lambda_k(s) = \frac{1}{2\pi^{\frac{k}{2}}}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s-k+1}{2}\right)}\Lambda(s)\Lambda(s-k+1).$$

Since $k$ is even, by repeated used of the identity $\Gamma(s+1) = s\Gamma(s)$ we get

$$\Gamma\left(\frac{s+1}{2}\right) = \Gamma\left(\frac{s-k+1}{2}+\frac{k}{2}\right) = \Gamma\left(\frac{s-k+1}{2}\right)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{s-k+1}{2}+n\right).$$

Thus

$$\Lambda_k(s) = \frac{1}{2\pi^{\frac{k}{2}}}\Lambda(s)\Lambda(s-k+1)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{s-k+1}{2}+n\right).$$

In Lecture 10 we learned that $\Lambda(s)$ has meromorphic continuation to $\mathbb{C}$ satisfying the functional equation $\Lambda(1-s) = \Lambda(s)$. It follows that $\Lambda_k(s)$ has meromorphic continuation to $\mathbb{C}$ and

$$\begin{aligned}
\Lambda_k(k-s) &= \frac{1}{2\pi^{\frac{k}{2}}}\Lambda(k-s)\Lambda(1-s)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{1-s}{2}+n\right)\\
&= \frac{1}{2\pi^{\frac{k}{2}}}\Lambda(1-(k-s))\Lambda(1-(1-s))\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{1-s}{2}+n\right)\\
&= \frac{1}{2\pi^{\frac{k}{2}}}\Lambda(s-k+1)\Lambda(s)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{1-s}{2}+\left(\frac{k}{2}-1-n\right)\right)\\
&= \frac{1}{2\pi^{\frac{k}{2}}}\Lambda(s-k+1)\Lambda(s)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{k-1-s}{2}-n\right)\\
&= (-1)^{\frac{k}{2}}\frac{1}{2\pi^{\frac{k}{2}}}\Lambda(s-k+1)\Lambda(s)\prod_{n=0}^{\frac{k}{2}-1}\left(\frac{s-k+1}{2}+n\right)\\
&= (-1)^{\frac{k}{2}}\Lambda_k(s).
\end{aligned}$$

**b**. We know that $\Lambda(s)$ has simple poles at $s=0$ and $s=1$, and it is holomorphic in $\mathbb{C}\setminus\{0,1\}$. It follows that $\Lambda_k(s)$ is holomorphic in $\mathbb{C}\setminus\{0,1,k-1,k\}$. Around $s=1$ the product $\Lambda(s)\Lambda(s-k+1)$ has a simple pole but the factor $\left(\frac{s-k+1}{2}+n\right)$ with $n=\frac{k}{2}-1$ vanishes at $s=1$, hence $\Lambda_k(s)$ is holomorphic at $s=1$. Similarly, around $s=k-1$ the product $\Lambda(s)\Lambda(s-k+1)$ has a simple pole but the factor $\left(\frac{s-k+1}{2}+n\right)$ with $n=0$ vanishes at $s=k-1$, hence $\Lambda_k(s)$ is holomorphic at $s=k-1$. This proves that $\Lambda_k(s)$ is holomorphic in $\mathbb{C}\setminus\{0,k\}$ with simple p[oles at $s=0$ and $s=k$.

**c**. We have

$$\begin{aligned}
L_k(s) &= \zeta(s)\zeta(s-k+1)\\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{m=1}^{\infty}\frac{1}{m^{s-k+1}}\\
&= \sum_{n,m=1}^{\infty}\frac{m^{k-1}}{(nm)^s}\\
&= \sum_{N=1}^{\infty}\frac{\sigma_{k-1}(N)}{N^s}.
\end{aligned}$$

Since $k\geq 4$, we can consider the modular form

$$\left(-\frac{B_k}{2k}\right)E_k(\tau) = -\frac{B_k}{2k}+\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where $q=e^{2\pi i\tau}, \tau\in\mathbb{H}$. Then, $\Lambda_k(s)$ is the $L$-function associated to this modular form. In the case $k=2$ the function $\Lambda_k(s)$ is not the $L$-function of a modular form of weight 2, since $M_2=\{0\}$. Note that $\Lambda_2(s)=\frac{1}{2\pi}\Lambda(s)\Lambda(s-1)\left(\frac{s-1}{2}\right)$ has simple poles at $s=0,1,2$, hence it does not satisfy the analytic properties of a modular $L$-function.

**3**. **a**. Note that by multiplicativity we have $a_1=a_1a_1$, hence $a_1=1$. Now, given a positive integer $M$ define

$$P_M := \{p\in\mathbb{Z}^+ : p \text{ prime}, p\leq M\},$$

and

$$I_M := \{n\in\mathbb{Z}^+ : \text{ all prime divisors of } n \text{ are in } P_M\}.$$

Since the series $\sum a_n$ converges absolutely, the series

$$S_p := \sum_{k=0}^{\infty}a_{p^k} = 1+a_p+a_{p^2}+\dots$$

also converges absolutely. Now, since $(a_n)_{n\in\mathbb{N}}$ is multiplicative, for any $M\geq 2$ we have

$$\prod_{p\in P_M}S_p = \sum_{n\in I_M}a_n.$$

If we define $q_M$ as the first prime number with $q_M > M$ (which is also the first positive integer larger than $M$ that has at least one prime divisor not in $P_M$), then

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n\in I_M} a_n\right| \leq \sum_{n\geq q_M} |a_n| \to_{M\to\infty} 0, \text{ since } q_M \to \infty.$$

This implies

$$\prod_p \left(1 + a_p + a_{p^2} + \ldots\right) = \lim_{M\to\infty} \prod_{p\in P_M} S_p = \lim_{M\to\infty} \sum_{n\in I_M} a_n = \sum_{n=1}^{\infty} a_n,$$

as desired. Finally, we note that the infinite product

$$\prod_p \left(1 + a_p + a_{p^2} + \ldots\right)$$

converges absolutely since

$$\sum_p \left|a_p + a_{p^2} + \ldots\right| \leq \sum_{n=2}^{\infty} |a_n| < \infty$$

(see *Complex Analysis* of S. Lang (Springer 1999), chapter XIII, Lemma 1.1).

**b.** For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ put $a_n(s) := n^{-s}$. Clearly, $(a_n(s))_{n\in\mathbb{N}}$ is multiplicative with $a_1(s) = 1$. Since the series $\zeta(s) = \sum a_n(s)$ converges absolutely, we have

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 + a_p(s) + a_{p^2}(s) + \ldots\right) \\ &= \prod_p \left(1 + p^{-s} + p^{-2s} + \ldots\right) \\ &= \prod_p \left(1 - p^{-s}\right)^{-1}, \end{aligned}$$

where in the last equality one uses $1 + r + r^2 + \ldots = (1-r)^{-1}$ for $r \in \mathbb{C}$ with $|r| < 1$, choosing $r = p^{-s}$.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > k$ we get

$$\zeta(s-k+1) = \prod_p \left(1 - p^{k-1-s}\right)^{-1}$$

hence

$$L_k(s) = \prod_p \left((1-p^{-s})(1-p^{k-1-s})\right)^{-1}.$$

4. **a.** By the dimension formulas from Lecture 6 we know that $S_{24}$ has dimension 2, so it is enough to prove that $f_1$ and $f_2$ are linearly independent. But $\frac{f_1}{f_2} = \frac{\Delta}{E_6^2}$ is not constant since $\Delta$ is cuspidal and $E_6^2$ is not cuspidal. Hence, $f_1, f_2$ are linearly independent.

**b.** We have

$$\begin{aligned} \Delta(\tau) &= q - 24q^2 + 252q^3 - 1472q^4 + \ldots, \\ E_6(\tau) &= 1 - 504q - 16632q^2 - 122976q^3 + \ldots, \end{aligned}$$

hence

$$\begin{aligned} f_1(\tau) &= q^2 - 48q^3 + 1080q^4 + \ldots, \\ f_2(\tau) &= q - 1032q^2 + 245196q^3 + 10965568q^4 + \ldots. \end{aligned}$$

If we write $f_1 = \sum_{n=1}^{\infty} a_n q^n$, $f_2 = \sum_{n=1}^{\infty} b_n q^n$ $T_2(f_1) = \sum_{n=1}^{\infty} c_n q^n$ and $T_2(f_2) = \sum_{n=1}^{\infty} d_n q^n$ then

$$\begin{aligned} c_n &= \sum_{t|(2,n)} t^{23} a\left(\frac{2n}{t^2}\right), \\ d_n &= \sum_{t|(2,n)} t^{23} b\left(\frac{2n}{t^2}\right). \end{aligned}$$

In particular

$$\begin{aligned} c_1 &= a_2 = 1, \\ c_2 &= a_4 + 2^{23} a_1 = 1080, \\ d_1 &= b_2 = -1032, \\ d_2 &= b_4 + 2^{23} b_1 = 10965568 + 2^{23} = 19354176. \end{aligned}$$

It follows that

$$\begin{aligned} T_2(f_1) &= q + 1080q^2 + \ldots = f_2 + 2112 f_1, \\ T_2(f_2) &= -1032q + 19354176q^2 + \ldots = -1032 f_2 + 18289152 f_1. \end{aligned}$$

Hence the matrix of $T_2$ in the basis $\{f_1, f_2\}$ is

$$\begin{pmatrix} 2112 & 18289152 \\ 1 & -1032 \end{pmatrix}.$$

**c**. The eigenvector of the above matrix are $(12(131 + \sqrt{144169}), 1)$ and $(12(131 - \sqrt{144169}), 1)$, with eigenvalues $\lambda_1 = 12(45 + \sqrt{144169})$ and $\lambda_2 = 12(45 - \sqrt{144169})$, respectively. It follows that

$$\begin{aligned} F_1 &= 12(131 + \sqrt{144169}) f_1 + f_2 = q + 12(45 + \sqrt{144169}) q^2 + \ldots, \\ F_2 &= 12(131 - \sqrt{144169}) f_1 + f_2 = q + 12(45 - \sqrt{144169}) q^2 + \ldots, \end{aligned}$$

form a basis for $S_{24}$ consisting of normalized eigenforms for $T_2$. Since all the other Hecke operators commute with $T_2$ and $F_1, F_2$ are in eigenspaces for $T_2$ of dimension one, it follows that they are also eigenforms for all the other Hecke operators.