Solutions Sheet 5

1. By definition (see Lecture 4) we have $E_k = \frac{1}{2\zeta(k)} G_k$ where

$$G_k(z) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}.$$

Writing $(m,n) = t(c,d)$ with $t := \text{g.c.d.}(m,n)$ and $c,d \in \mathbb{Z}$ coprimes, we have

$$G_k(z) = \sum_{t=1}^{\infty} \frac{1}{t^k} \sum_{\substack{c,c\in\mathbb{Z} \\ \text{g.c.d.}(c,d)=1}} \frac{1}{(cz+d)^k} = \zeta(k) \sum_{\substack{c,c\in\mathbb{Z} \\ \text{g.c.d.}(c,d)=1}} \frac{1}{(cz+d)^k},$$

hence

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,c\in\mathbb{Z} \\ \text{g.c.d.}(c,d)=1}} \frac{1}{(cz+d)^k}. \tag{1}$$

Now, define $P = \{(c,d) \in \mathbb{Z}^2 : \text{g.c.d.}(c,d) = 1\}$ and consider the equivalence relation in P given by

$$(c,d) \sim (c',d') \Leftrightarrow \Big((c,d) = (c',d') \text{ or } (c,d) = -(c',d')\Big).$$

Then the map $\Gamma_\infty \backslash \Gamma \to P/\sim$ defined by

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d)$$

is a bijection. This implies

$$\begin{aligned} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} 1|_k\gamma &= \sum_{(c,d)\in P/\sim} 1|_k \begin{pmatrix} * & * \\ c & d \end{pmatrix} \\ &= \sum_{(c,d)\in P/\sim} (cz+d)^{-k} \\ &= \frac{1}{2} \sum_{(c,d)\in P} (cz+d)^{-k}. \end{aligned}$$

By (1) this equals E_k as claimed.
Now, let F be a fundamental domain for Γ. Given $g \in S_k$ we compute

$$\begin{aligned} \langle E_k, g\rangle &= \int_F E_k(z)\overline{g(z)} y^k d\mu(z) \\ &= \int_F \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} 1|_k\gamma \overline{g(z)} y^k d\mu(z) \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_F \overline{g(\gamma z)} \text{Im}(\gamma z)^k d\mu(z) \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma F} \overline{g(z)} \text{Im}(z)^k d\mu(z) \\ &= \int_{F_\infty} \overline{g(z)} \text{Im}(z)^k d\mu(z) \end{aligned}$$

where

$$F_\infty := \bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma} \gamma F.$$

Note that F_∞ is a fundamental domain for Γ_∞ (see Exercise Class 1). Since the integral above is independent of the choice of F_∞ we can take $F_\infty = [0,1]\times]0,\infty[$. Writing

$$g(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

we get

$$\langle E_k, g\rangle = \int_0^\infty \int_0^1 \overline{g(z)} y^{k-2} dx dy = \sum_{n=1}^{\infty} \overline{a_n} \int_0^\infty y^{k-2} e^{-2\pi n y} \int_0^1 e^{-2\pi i n x} dx dy.$$

But

$$\int_0^1 e^{-2\pi i n x} dx = \frac{e^{-2\pi i n x}}{-2\pi i n}\Big|_{x=0}^{x=1} = 0 \text{ for all } n \neq 0.$$

This implies $\langle E_k, g\rangle = 0$.

2. On the one hand, since $P_n \in S_k$ for every $n \geq 1$, we have $\langle E_k, P_n\rangle = 0$ by **1**. On the other hand, the n-th Fourier coefficient of E_k is

$$a_n = -\frac{2k}{B_k}\sigma_{k-1}(n).$$

Since $\sigma_{k-1}(n) \geq 1$ for all n, we have $a_n \neq 0$. This implies that the formula is not valid.

Added comment: On can check that the proof of Theorem 5.4 in Lecture 11 is not valid when $f = E_k$ due to convergence issues (see Exercise Class 5).

3. **a**. By definition of congruence subgroup we have $\Gamma(N_1) \subseteq \Gamma'$ and $\Gamma(N_2) \subseteq \Gamma''$ for some pair of positive integers N_1, N_2, where $\Gamma(N)$ denotes the principal congruence subgroup of level N (see Lecture 4). Thus, we have

$$\Gamma' \cap \Gamma'' \supseteq \Gamma(N_1) \cap \Gamma(N_2) \supseteq \Gamma(N_1 N_2).$$

It follows that $\Gamma' \cap \Gamma''$ is also a congruence subgroup. Hence, in order to prove that

$$\langle f, g\rangle_{\Gamma''} = \langle f, g\rangle_{\Gamma'} \text{ for all } f, g \in S_k(\Gamma') \cap S_k(\Gamma''),$$

we can assume $\Gamma'' \subseteq \Gamma'$ (by replacing Γ'' by $\Gamma' \cap \Gamma''$ if necessary). Now, let F' be a fundamental domain for Γ' and denote by $\gamma \mapsto \overline{\gamma}$ the natural projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$. If $\gamma_1, \ldots, \gamma_m$ are elements of Γ' such that

$$\overline{\Gamma'} = \bigsqcup_{i=1}^{m} \overline{\Gamma''}\,\overline{\gamma_i},$$

then

$$\mu(\gamma_i F' \cap \gamma_j F') = 0 \text{ for all } i, j \in \{1, \ldots, m\} \text{ with } i \neq j,$$

and

$$F'' := \bigcup_{i=1}^{m} \gamma_i F'$$

is a fundamental domain for Γ''. Hence, for $f, g \in S_k(\Gamma'')$ we have

$$\begin{aligned}
\langle f, g\rangle_{\Gamma''} &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma''}]} \int_{\Gamma''\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) \\
&= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}][\overline{\Gamma'} : \overline{\Gamma''}]} \int_{F''} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) \\
&= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]m} \sum_{i=1}^{m} \int_{\gamma_i F'} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) \\
&= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]m} \sum_{i=1}^{m} \int_{F'} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z),
\end{aligned}$$

since the integrand is Γ' invariant. Thus

$$\langle f, g\rangle_{\Gamma''} = \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]} \int_{F'} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) = \langle f, g\rangle_{\Gamma'}.$$

b. Let us write $\alpha = \frac{1}{m}\alpha'$ where $m \geq 1$ is an integer and $\alpha' \in \mathrm{M}_{2\times 2}(\mathbb{Z})$. Then $\alpha^{-1}\Gamma\alpha = (\alpha')^{-1}\Gamma\alpha'$, so we can assume that α has integer coefficients. Now, let D be the determinant of α. Then D is a positive integer and $\alpha\Gamma(D)\alpha^{-1} \subseteq \Gamma$. Indeed, a matrix in $\Gamma(D)$ is of the form $\gamma = I + DM$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M \in M_{2\times 2}(\mathbb{Z})$, so

$$\alpha\gamma\alpha^{-1} = I + \alpha M(D\alpha^{-1}) \in M_{2\times 2}(\mathbb{Z}).$$

Since $\det(\alpha\gamma\alpha^{-1}) = \det(\gamma) = 1$, we have $\alpha\Gamma(D)\alpha^{-1} \subseteq \Gamma$.
It follows that $\Gamma' \supseteq \Gamma(D)$, thus Γ' is a congruence subgroup of Γ.

c. It is clear that $f|_k\alpha, g|_k\alpha \in S_k(\Gamma')$. Now, let F' be a fundamental domain for Γ'. Then

$$\begin{aligned}\langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'} &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]}\int_{F'} f|_k\alpha(z)\overline{g|_k\alpha(z)}\mathrm{Im}(z)^k d\mu(z) \\ &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]}\int_{F'} f(\alpha z)\overline{g(\alpha z)}\mathrm{Im}(\alpha z)^k d\mu(z) \\ &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}]}\int_{\alpha F'} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z).\end{aligned}$$

The set $\alpha F'$ is a fundamental domain for $\Gamma'' := \alpha\Gamma'\alpha^{-1} = \alpha\Gamma\alpha^{-1} \cap \Gamma$, hence by **a.** we have

$$\langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'} = \frac{[\overline{\Gamma} : \overline{\Gamma''}]}{[\overline{\Gamma} : \overline{\Gamma'}]}\langle f, g\rangle_{\Gamma''} = \frac{[\overline{\Gamma} : \overline{\Gamma''}]}{[\overline{\Gamma} : \overline{\Gamma'}]}\langle f, g\rangle_{\Gamma}.$$

Now, for fundamental domains F, F' and F'' for Γ, Γ' and Γ'', respectively, we have

$$\mu(F'') = \mu(F)[\overline{\Gamma} : \overline{\Gamma''}],\ \mu(F') = \mu(F)[\overline{\Gamma} : \overline{\Gamma'}].$$

Since we can take $F'' = \alpha F'$, and $\mu(F') = \mu(\alpha F')$, we get $[\overline{\Gamma} : \overline{\Gamma'}] = [\overline{\Gamma} : \overline{\Gamma''}]$. We conclude $\langle f|_k\alpha, g|_k\alpha\rangle_{\Gamma'} = \langle f, g\rangle_{\Gamma}$, as desired.

Added comment: We showed above, comparing measures of fundamental domains, that for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ we have $[\overline{\Gamma} : \overline{\Gamma'}] = [\overline{\Gamma} : \overline{\Gamma''}]$ where $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma'' = \Gamma \cap \alpha\Gamma\alpha^{-1}$. This is equivalent to the equality $[\Gamma : \Gamma'] = [\Gamma : \Gamma'']$, which in turn can be proved using properties of double cosets. Indeed, first note that the map $\gamma \mapsto \gamma\alpha^{-1}$ induces a bijection $\Gamma\backslash\Gamma\alpha\Gamma \to \Gamma\backslash(\Gamma\alpha\Gamma\alpha^{-1})$, hence

$$\begin{aligned}\#\Gamma\backslash\Gamma\alpha\Gamma &= \#\Gamma\backslash(\Gamma\alpha\Gamma\alpha^{-1}) \\ &= \#\Gamma\backslash(\Gamma\cdot\alpha\Gamma\alpha^{-1}) \\ &= \#(\Gamma\cap\alpha\Gamma\alpha^{-1})\backslash\alpha\Gamma\alpha^{-1},\end{aligned}$$

where we have used tha canonical bijection $H\backslash HK \to (H \cap K)\backslash K$ valid for any pair of subgroups H, K in some group G. Conjugating by α we get

$$\#(\Gamma\cap\alpha\Gamma\alpha^{-1})\backslash\alpha\Gamma\alpha^{-1} = \#(\Gamma\cap\alpha^{-1}\Gamma\alpha)\backslash\Gamma,$$

hence

$$\#(\Gamma\backslash\Gamma\alpha\Gamma) = [\Gamma : \Gamma\cap\alpha^{-1}\Gamma\alpha].$$

Similarly

$$\#(\Gamma\alpha\Gamma/\Gamma) = [\Gamma : \Gamma\cap\alpha\Gamma\alpha^{-1}].$$

Now, by Proposition 1.4.3 in Bump's book *Automorphic Forms and Representations* we have $\Gamma\alpha\Gamma = \Gamma\alpha^t\Gamma$, hence the map $\gamma \mapsto \gamma^t$ induces a bijection $\Gamma\backslash\Gamma\alpha\Gamma \to \Gamma\alpha\Gamma/\Gamma$ and we get

$$[\Gamma : \Gamma\cap\alpha^{-1}\Gamma\alpha] = [\Gamma : \Gamma\cap\alpha\Gamma\alpha^{-1}]$$

as desired.

d. Given $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ put $\alpha' := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \mathrm{del}(\alpha)\alpha^{-1}$. Then $f|_k(\alpha^{-1}) = f|_k\alpha'$ for any function $f : \mathbb{H} \to \mathbb{C}$. If follows from **c.**, by replacing g by $g|_k(\alpha^{-1})$, that

$$\langle f|_k\alpha, g\rangle_G = \langle f, g|_k\alpha'\rangle_G, \tag{2}$$

where G is any congruence subgroup of Γ contained in $\alpha^{-1}\Gamma\alpha \cap (\alpha')^{-1}\Gamma\alpha'$. Now, by definition we have

$$T_n(f) = n^{\frac{k}{2}-1}\sum_{\gamma\in\mathcal{R}(n)} f|_k\gamma$$

where $\mathcal{R}(n)$ is any set of representatives for $\Gamma\backslash M(n)$ with $M(n)$ the set of all 2 by 2 matrices with integer coefficients and determinant n. We can take

$$\mathcal{R}(n) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) : ad = n, a > 0, b \in Z_d \right\}$$

where Z_d is any set of representatives for $\mathbb{Z}/d\mathbb{Z}$. Now, by (2) we have, for every $f, g \in S_k(\Gamma)$ the equalities

$$\langle T_n(f), g\rangle_\Gamma = \langle T_n(f), g\rangle_G = \left\langle f, n^{\frac{k}{2}-1} \sum_{\gamma\in\mathcal{R}(n)} g|_k\gamma' \right\rangle_G$$

where G is any congruence subgroup of Γ contained in the intersection of all the groups $\gamma^{-1}\Gamma\gamma \cap (\gamma')^{-1}\Gamma\gamma'$ with γ in a set of representatives $\mathcal{R}(n)$. But the set $\{\gamma' : \gamma \in \mathcal{R}(n)\}$ is

$$\left\{ \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) : ad = n, a > 0, b \in Z_d \right\},$$

which is just another set of representatives for $\Gamma\backslash M(n)$, thus

$$\langle T_n(f), g\rangle_\Gamma = \langle f, T_n(g)\rangle_G = \langle f, T_n(g)\rangle_\Gamma.$$

This proves that T_n is self-adjoint.

4. **a**. Since $1 \le j \le d$ we have that f_j is holomorphic in $\mathbb{H}$. The weight of f_j is

$$12j + 6(2(d-j) + b) + 4a = 12d + 6b + 4a.$$

If $k \not\equiv 2 \pmod{12}$, then $d = \lfloor \frac{k}{12} \rfloor$ and $4a + 6b \in \{0, 4, 6, 8, 10\}$, thus $k = 12t + r$ with $r := 4a + 6b$ and $t \ge 0$ an integer. It follows that $d = t$ and by the above computation the weight of f_j is k. If $k \equiv 2 \pmod{12}$, then $d = \lfloor \frac{k}{12} \rfloor - 1$ and $4a + 6b = 14$, thus $k = 12t + r$ with $r := 2$ and $t \ge 0$ some integer. Hence $d = t - 1$, $r = 4a + 6b - 12$ and by the above computation the weight of f_j is k. This proves that $f_j \in M_k$.
Since $f_j \in \Delta M_{k-12}$ it follows that f_j is cuspidal. The statements on the Fourier coefficients $a_n^{(j)}$ follow from the fact that Δ, E_4 and E_6 have integer Fourier coefficients with first terms $q, 1$ and 1, respectively.

b. The forms in $\{f_1, \ldots, f_d\}$ are clearly linearly independent. Since it contains $d = \dim_{\mathbb{C}}(S_k)$ forms, it is a basis of S_k.

c. If $g \in S_k$ is a $\mathbb{Z}$-linear combination of $\{f_1, \ldots, f_d\}$, then it has integral Fourier coefficients. Conversely, if g has Fourier coefficients c_n $(n \ge 1)$, then

$$g = \sum_{i=1}^{d} \alpha_i f_i,$$

where the coefficients α_i satisfy the recurrence formula

$$\begin{aligned} \alpha_1 &= c_1, \\ \alpha_i + \sum_{j=1}^{i-1} \alpha_j a_i^{(j)} &= c_i \text{ for } i \in \{2, \ldots, d\}. \end{aligned}$$

Since the coefficients c_n are integers, it follows by recursion that $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ as desired.