Solutions Sheet 6

1. Given $x \in X$ define $Y_x := \{y \in Y : (x, y) \in S\}$. First we have to show that Y_x is G_x-invariant. For $(x, y) \in Y_x$ and $g \in G_x$ we have $g \cdot (x, y) = (x, g \cdot y)$. Since S is invariant under G and $(x, y) \in S$ we have $(x, g \cdot y) = g \cdot (x, y) \in S$. Hence $g \cdot y \in Y_x$ as desired.
Let's now check that the quantity

$$n(x) := \#(G_x \backslash Y_x)$$

is G-invariant, i.e. $n(g \cdot x) = n(x)$ for all $g \in G$, so that $n(x)$ is well defined for $x \in G\backslash X$. To do this note first that $G_{g\cdot x} = gG_x g^{-1}$ and $Y_{g\cdot x} = g \cdot Y_x$. It follows that the map $y \mapsto g \cdot y$ induces a bijection

$$G_x \backslash Y_x \mapsto G_{g\cdot x} \backslash Y_{g\cdot x}.$$

(with inverse induced by $y \mapsto g^{-1} \cdot y$). This implies $n(x) = n(g \cdot x)$ as claimed.
Now we prove the equality

$$\#(G\backslash S) = \sum_{x\in G\backslash X} \#(G_x\backslash Y_x) = \sum_{x\in G\backslash X} n(x). \tag{1}$$

Let $\pi_X : S \to X$ denote the projection on the first coordinate. It induces a well-defined map $\overline{\pi}_X : G\backslash S \to G\backslash X$ and

$$|G\backslash X| = \sum_{x\in G\backslash X} |\overline{\pi}_X^{-1}(x)|. \tag{2}$$

Given a representative x_0 of $x \in G\backslash X$ it is easy to see that $y \mapsto (x_0, y)$ induces a bijection

$$G_{x_0}\backslash Y_{x_0} \to \overline{\pi}_X^{-1}(x).$$

Hence $n(x) = n(x_0) = |\overline{\pi}_X^{-1}(x)|$ and by (2) we get the desired formula (1).

Comment: Formula (1) was used in the proof of Theorem 7.7 in Lecture 18.

2. We will use Poisson summation. Recall that given a Schwarz function $f : \mathbb{R} \to \mathbb{C}$ one has the identity

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

where $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of f defined as

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx.$$

In the case $f(x) = f_{a,t}(x) = e^{-\pi(x+a)^2 t}$, where $a, t \in \mathbb{R}$ with $t > 0$ one has $\widehat{f_{a,t}}(y) = \frac{1}{\sqrt{t}}e^{2\pi iay} f_{0,1/t}(y)$.
Similarly, in the case $g(x) = g_{a,t}(x) = e^{-\pi(x+ia)^2 t}$, where $a, t \in \mathbb{R}$ with $t > 0$ one has $\widehat{g_{a,t}}(y) = \frac{1}{\sqrt{t}}e^{-2\pi ay} g_{0,1/t}(y)$.
Now, writing $z = x + iy$ we have

$$\begin{aligned}
\Theta_z(t) &= \sum_{m,n\in\mathbb{Z}} e^{-\pi t\frac{|mz+n|^2}{y}} \\
&= \sum_{m,n\in\mathbb{Z}} e^{-\pi t\frac{(mx+n)^2+(my)^2}{y}} \\
&= \sum_{m\in\mathbb{Z}} e^{-\pi tm^2 y} \sum_{n\in\mathbb{Z}} e^{-\pi(mx+n)^2\frac{t}{y}} \\
&= \sum_{m\in\mathbb{Z}} e^{-\pi tm^2 y} \sum_{n\in\mathbb{Z}} f_{mx,t/y}(n).
\end{aligned}$$

By Poisson summation we get

$$\begin{aligned}
\Theta_z(t) &= \sum_{m\in\mathbb{Z}} e^{-\pi t m^2 y} \sum_{n\in\mathbb{Z}} \sqrt{\frac{y}{t}} e^{2\pi i m x n} f_{0,y/t}(n) \\
&= \sum_{m\in\mathbb{Z}} e^{-\pi t m^2 y} \sum_{n\in\mathbb{Z}} \sqrt{\frac{y}{t}} e^{2\pi i m x n} e^{-\pi \frac{y}{t} n^2} \\
&= \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{y}{t} n^2} \sum_{m\in\mathbb{Z}} e^{-\pi t m^2 y} e^{2\pi i m x n} \\
&= \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{x^2+y^2}{ty} n^2} \sum_{m\in\mathbb{Z}} e^{-\pi t y \left(m - \frac{inx}{ty}\right)^2} \\
&= \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{x^2+y^2}{ty} n^2} \sum_{m\in\mathbb{Z}} g_{-nx/(ty),ty}(m).
\end{aligned}$$

Applying Poisson summation once more we get

$$\begin{aligned}
\Theta_z(t) &= \sqrt{\frac{y}{t}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{x^2+y^2}{ty} n^2} \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{ty}} e^{2\pi \frac{nx}{ty} m} g_{0,1/(ty)}(m) \\
&= \frac{1}{t} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{x^2+y^2}{ty} n^2} \sum_{m\in\mathbb{Z}} e^{2\pi \frac{nx}{ty} m} e^{-\pi \frac{m^2}{ty}} \\
&= \frac{1}{t} \sum_{n,m\in\mathbb{Z}} e^{-\pi \frac{(xn-m)^2+y^2n^2}{ty}} \\
&= \frac{1}{t} \sum_{n,m\in\mathbb{Z}} e^{-\pi \frac{|nz-m|^2}{ty}} \\
&= \frac{1}{t} \sum_{n,m\in\mathbb{Z}} e^{-\pi \frac{|mz+n|^2}{ty}} \\
&= \frac{1}{t} \Theta_z\left(\frac{1}{t}\right).
\end{aligned}$$

3. **a**. The fact that $E(\gamma z, s) = E(\gamma, s)$ for all $\gamma \in \Gamma$ follows from the first definition. Indeed, if $\mathcal{R} \subseteq \Gamma$ is a set of representatives for $\Gamma_\infty\backslash\Gamma$ and $\gamma \in \Gamma$, then

$$E(\gamma z, s) = \sum_{g\in\mathcal{R}} \mathrm{Im}(g\gamma z) = \sum_{g\in\mathcal{R}\gamma} \mathrm{Im}(gz).$$

But $\mathcal{R}\gamma$ is also a set of representatives for $\Gamma_\infty\backslash\Gamma$, hence this equals $E(z,s)$ as wanted. Now, we compute

$$\begin{aligned}
\int_0^\infty (\Theta_z(t) - 1) t^s \frac{dt}{t} &= \int_0^\infty \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} e^{-\pi t \frac{|mz+n|^2}{y}} t^s \frac{dt}{t} \\
&= \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \int_0^\infty e^{-\pi t \frac{|mz+n|^2}{y}} t^s \frac{dt}{t} \\
&= \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \left(\pi \frac{|mz+n|^2}{y}\right)^{-s} \int_0^\infty e^{-t} t^s \frac{dt}{t} \\
&= \pi^{-s}\Gamma(s) \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}} \\
&= E^*(z,s).
\end{aligned}$$

b. Using part **a**. we can write

$$\begin{aligned} E^*(z,s) &= \int_0^\infty (\Theta_z(t)-1)t^s \frac{dt}{t} \\ &= \int_1^\infty (\Theta_z(t)-1)t^s \frac{dt}{t} + \int_0^1 (\Theta_z(t)-1)t^s \frac{dt}{t}. \end{aligned}$$

Now, using **2**. we write

$$\begin{aligned} \int_0^1 (\Theta_z(t)-1)t^s \frac{dt}{t} &= \int_0^1 \Theta_z(t)t^s \frac{dt}{t} - \frac{t^s}{s}\Big|_{t=0}^{t=1} \\ &= \int_0^1 \Theta_z\left(\frac{1}{t}\right) t^{s-1}\frac{dt}{t} - \frac{1}{s} \\ &= \int_1^\infty \Theta_z\left(t\right) t^{1-s}\frac{dt}{t} - \frac{1}{s} \\ &= \int_1^\infty (\Theta_z\left(t\right)-1) t^{1-s}\frac{dt}{t} + \int_1^\infty t^{1-s}\frac{dt}{t} - \frac{1}{s} \\ &= \int_1^\infty (\Theta_z\left(t\right)-1) t^{1-s}\frac{dt}{t} + \frac{t^{1-s}}{1-s}\Big|_{t=1}^{t=\infty} - \frac{1}{s} \\ &= \int_1^\infty (\Theta_z\left(t\right)-1) t^{1-s}\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}. \end{aligned}$$

We get

$$E^*(z,s) = \int_1^\infty (\Theta_z(t)-1)t^s \frac{dt}{t} + \int_1^\infty (\Theta_z\left(t\right)-1) t^{1-s}\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}.$$

Since the function $t \mapsto \Theta_z(t)-1$ decays exponentially as $t \to \infty$, we conclude that

$$s \in \mathbb{C} \to \int_1^\infty (\Theta_z(t)-1)t^s \frac{dt}{t}$$

is entire. Hence, $E^*(z,s)$ has meromorphic continuation to $s \in \mathbb{C}$ with singularities only at $s = 0, 1$, which are simple poles with residues -1 and 1, respectively. Moreover, the above formula also shows that $E^*(z,s) = E^*(z, 1-s)$.

4. From Theorem 7.7 in Lecture 18 we know that

$$r_D(2^e) = \#\{b \in \mathbb{Z}/2^{e+1}\mathbb{Z} : b^2 \equiv D \,(\text{mod } 2^{e+2})\}.$$

• Assume $D \equiv 0 \pmod 4$ and $e \geq 2$. Then $D = 4d$ with $d \equiv 2, 3 \pmod 4$ squarefree. Now, any $b \in \mathbb{Z}/2^{e+1}\mathbb{Z}$ with $b^2 \equiv D \pmod{2^{e+2}}$ is of the form $b = 2b_0$ with $b \in \mathbb{Z}/2^e\mathbb{Z}$ and $b_0^2 \equiv d \pmod{2^e}$. Since $e \geq 2$ we have $b_0^2 \equiv d \equiv 2, 3 \pmod 4$. But this is impossible since 2 and 3 are not squares mod 4, so $r_D(2^e) = 0$ in this case, as claimed.
• Assume $D \equiv 0 \pmod 4$ and $e = 1$. Then

$$r_D(2) = \#\{b \in \mathbb{Z}/4\mathbb{Z} : b^2 \equiv D \,(\text{mod } 8\}.$$

The squares of $0, 1, 2, 3 \in \mathbb{Z}/4\mathbb{Z}$ are $0, 1, 4, 1 \in \mathbb{Z}/8\mathbb{Z}$, respectively. Hence

$$r_D(2) = \begin{cases} 1 & \text{if } D = 4d, d \equiv 2 \,(\text{mod } 4), \\ 1 & \text{if } D = 4d, d \equiv 3 \,(\text{mod } 4). \end{cases}$$

This coincides with $1 + \chi_D(2) = 1$.
• Assume $D \equiv 5 \pmod 8$ and $e \geq 1$. Since $\chi_D(2) = -1$ we have to show that $r_D(2^e) = 0$. If $b \in \mathbb{Z}/2^{e+1}\mathbb{Z}$ satisfies $b^2 \equiv D \pmod{2^{e+2}}$ then $b^2 \equiv 5 \pmod 8$. However, 5 is not a square mod 8, hence $r_D(2^e) = 0$ as waned.
• Finally, assume Assume $D \equiv 1 \pmod 8$ and $e \geq 1$. Since $\chi_D(2) = 1$ we have to show that $r_D(2^e) = 2$. We first show that $r_D(2^e) \leq 2$. Indeed, if $b_1, b_2 \in \mathbb{Z}/2^{e+1}\mathbb{Z}$ are solutions of $x^2 \equiv D \pmod{2^{e+2}}$, then

$$b_1^2 - b_2^2 = (b_1 - b_2)(b_1 + b_2) \equiv 0 \,(\text{mod } 2^{e+2}).$$

If $b_1 \not\equiv \pm b_2 \pmod{2^{e+1}}$ then we can write $b_1 - b_2 = 2^a u, b_1 + b_2 = 2^b v$ with u, v odd integers and a, b non-negative integers with $a + b \geq e + 2$. Replacing b_2 by $-b_2$ if necessary we can assume $a \leq b$. But $2b_1 = 2^a u + 2^b v$, hence $a \geq 1$ and $b_1 = 2^{a-1}u + 2^{b-1}v$. Since $D \equiv 1 \pmod 8$ implies b_1 odd, we conclude $a = 1$, hence $b \geq e + 1$. Thus $b_1 = -b_2 \pmod{2^{e+1}}$ which is a contradiction. This proves that $r_D(2^e) \leq 2$. In order to prove that $r_D(2^e) = 2$ we use induction on e. For $e = 1$ we have

$$r_D(2) = \#\{b \in \mathbb{Z}/4\mathbb{Z} : b^2 \equiv 1 \,(\text{mod } 8)\} = 2.$$

Now, assume $r_D(2^e) = 2$ and let b be an integer that is a solution of $x^2 \equiv D \pmod{2^{e+2}}$ (there are exactly two possible choices of b mod 2^{e+1}). Given $t \in \mathbb{Z}\backslash 2\mathbb{Z}$ the element $y_t := b + 2^{e+1}t \in \mathbb{Z}/2^{e+2}\mathbb{Z}$ satisfies

$$y_t^2 \equiv b^2 + 2^{e+2}bt \,(\text{mod } 2^{e+3}).$$

We know that $b^2 = D + n2^{e+2}$ for some integer n, thus

$$y_t^2 \equiv D + n2^{e+2} + 2^{e+2}bt = D + 2^{e+2}(n + bt) \,(\text{mod } 2^{e+3}).$$

Since b is odd (because D is) we have $b \equiv 1 \pmod 2$, so choosing $t \equiv -n \pmod 2$ gives $n + bt \equiv 0 \pmod 2$. This implies $y_t^2 \equiv D \pmod{2^{e+3}}$. We conclude that there exists $y \in \mathbb{Z}/2^{e+2}\mathbb{Z}$ solution of $x^2 \equiv D \pmod{2^{e+3}}$ satisfying also $y \equiv b \pmod{2^{e+1}}$. A different choice of b gives then another solution in $\mathbb{Z}/2^{e+2}\mathbb{Z}$ of $x^2 \equiv D \pmod{2^{e+3}}$. Since $r_D(2^e) = 2$ we get $r_D(2^{e+1}) \geq 2$. But we proved above that $r_D(2^{e+1}) \leq 2$, hence $r_D(2^{e+1}) = 2$ as claimed. This proves the result in the case $e \geq 1$ and $D \equiv 1 \pmod 8$.

Comment: It is nice to check this formula in a particular example. Choose $D = -4$. There is only one class of discriminant -4 and it is represented by the quadratic form $Q = [1, 0, 1]$ corresponding to $Q(x, y) = x^2 + y^2$, with $|\Gamma_Q| = 4$. According to the formula the we just proved, we have

$$r_{-4}(2^e) = \frac{1}{4} r_Q(2^e) = \begin{cases} 0 & \text{if } e \geq 2, \\ 1 & \text{if } e = 1, \end{cases}$$

since $\chi_{-4}(2) = 0$. In the case $e = 1$ we see that the primitive representations of 2 by Q are given by $\{(\pm 1, 1), (1, \pm 1)\}$, hence $r_Q(2) = 4$ as expected. For $e \geq 2$ we have that $x^2 + y^2$ does not represent primitively 2^e, since any such representation would have $x^2 + y^2 \equiv 0 \pmod 4$, but this implies that both x and y are even. Thus $r_Q(2^e) = 0$ in this case, as expected.