

Exercise Sheet 1

1. Let $G = \mathrm{SL}_2(\mathbb{R})$. The aim of this exercise is to show that $G_{\mathbb{Z}} = \mathrm{SL}_2(\mathbb{Z})$ is a lattice in G .

(a) Argue that $G_{\mathbb{Z}}$ is discrete in G and that both G and $G_{\mathbb{Z}}$ are unimodular.

From this we know that $G/G_{\mathbb{Z}}$ admits a nonzero G -invariant measure μ which is unique up to a non-zero constant. In order to show that $G_{\mathbb{Z}}$ is a lattice we have to show that $\mu(G/G_{\mathbb{Z}}) < \infty$. For this, we will use the following fact:

(b) Assume that there exists a measurable set $A \subseteq G$ of finite measure such that every $G_{\mathbb{Z}}$ -orbit intersects A , i.e. for every $g \in G$ there exists some $\gamma \in G_{\mathbb{Z}}$ such that $g\gamma \in A$. Show that $\mu(G/G_{\mathbb{Z}})$ is finite.

(c) Show that the map sending

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ to } z \mapsto g \cdot z := \frac{az + b}{cz + d}$$

is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \rightarrow \mathrm{Bih}(\mathbb{H}^2)$, where $\mathrm{Bih}(\mathbb{H}^2)$ denotes the biholomorphic maps of the complex upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Show that its kernel is $\{\pm I\}$ where I denotes as usual the 2×2 identity matrix.

(d) Prove that the induced homomorphism

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \{\pm I\} \rightarrow \mathrm{Bih}(\mathbb{H}^2)$$

of (c) is actually an isomorphism. For the action of $\mathrm{SL}_2(\mathbb{R})$ on \mathbb{H}^2 from above determine the orbit Gi and stabilizer K of $i \in \mathbb{H}^2$. (Show also that K is compact.) Using this, show that we have a diffeomorphism

$$G/K \longrightarrow \mathbb{H}^2, g \longmapsto g \cdot i.$$

(e) Set $K = \mathrm{SO}_2(\mathbb{R})$,

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R}, a > 0 \right\},$$

$$A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \mid y \in \mathbb{R}^+ \right\}, \text{ and}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P \times K \longrightarrow G, (p, k) \longmapsto pk$$

and

$$N \times A \longrightarrow P, (n, a) \longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms?

Show that P is a semidirect product $N \rtimes A$ and that we have the diffeomorphism $N \times A \cong \mathbb{H}^2$.

- (f) Prove that K is unimodular by showing that $d\mu_{\mathbb{H}^2} = y^{-2} dx dy$, $z = x + iy$, is a G -invariant volume form on the G -homogeneous space \mathbb{H}^2 .
- (g) Let

$$\mathcal{F} := \{z \in \mathbb{H}^2 \mid (|z| > 1 \text{ and } -1/2 \leq \operatorname{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \leq \operatorname{Re}(z) \leq 0)\}.$$

Show that for all $z \in \mathbb{H}^2$ the orbit $G_{\mathbb{Z}}z$ intersects \mathcal{F} in a unique point.

Hint: For every $G_{\mathbb{Z}}$ -orbit $G_{\mathbb{Z}}z$, $z \in \mathbb{H}^2$, consider $w \in G_{\mathbb{Z}}z$ with maximal imaginary part.

- (h) Show that the volume of \mathcal{F} with respect to $\mu_{\mathbb{H}^2}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

2. Consider the hyperbolic n -space

$$\mathbb{H}^n = \{p \in \mathbb{R}^{n+1} : b(p, p) = -1 \text{ and } p_{n+1} \geq 1\}$$

defined by the bilinear form $b(p, q) = p_1q_1 + \dots + p_nq_n - p_{n+1}q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : \begin{array}{l} \text{There exists a smooth path } \gamma: (-1, 1) \rightarrow \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot{\gamma}(0) = x \end{array} \right\}.$$

- (a) Show that $T_p\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : b(p, x) = 0\}$.
- (b) Show that $g_p = b|_{T_p\mathbb{H}^n} : T_p\mathbb{H}^n \times T_p\mathbb{H}^n \rightarrow \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p\mathbb{H}^n$. This means that g_p is a scalar product and (\mathbb{H}^n, g) is a Riemannian manifold.
Hint: Use (a) and the Cauchy-Schwarz-inequality on \mathbb{R}^n .
- (c) Show that the map $s_p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, $q \mapsto -2p \cdot b(p, q) - q$ defines a well defined geodesic symmetry of \mathbb{H}^n , i.e. it is an involution, with an isolated fixed point p . This means that the hyperbolic plane \mathbb{H}^n is a symmetric space.

3. Show that $A \mapsto gAg^t$ defines a group action of $\operatorname{SL}(n, \mathbb{R}) \ni g$ on

$$\mathcal{P}(n) = \{A \in M_{n \times n}(\mathbb{R}) : A = A^t, \det A = 1, A \gg 0\}.$$

Show that this action is transitive, i.e. $\forall A, B \in \mathcal{P}(n) \exists g \in \operatorname{SL}(n, \mathbb{R}) : gAg^t = B$. You may use the Linear-Algebra-fact that symmetric matrices are orthogonally diagonalizable, i.e. if $A = A^t$, then $\exists Q \in \operatorname{SO}(n, \mathbb{R})$ such that QAQ^t is diagonal.