Exercise Sheet 2

Exercise 1 (Compact Lie groups as symmetric spaces)**.** Let G be a compact connected Lie group and let

$$G^* = \{(g, g) \in G \times G : g \in G\} < G \times G$$

denote the diagonal subgroup.

(a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

(b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

(c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the *Riemannian* exponential map from $\mathfrak{g}$ into the Riemannian globally symmetric space G.

Exercise 2 (Compact semisimple Lie groups as symmetric spaces)**.** A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Exercise 3 (Closed differential forms)**.** Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\mathrm{Isom}(M)^\circ$. Prove that $d\omega = 0$.

Exercise 4 (A symmetric space with non-compact K)**.** Let $G = \widetilde{\mathrm{SL}(2, \mathbb{R})}$ and $K = \widetilde{\mathrm{SO}(2, \mathbb{R})}$. The aim of this exercise is to show that (G, K) is a symmetric pair with non-compact K.

(a) Prove that $\sigma\colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R}), g \mapsto {}^t g^{-1}$ is an involution.

(b) By covering space theory we can lift σ to the universal cover G. Prove that $\tilde{\sigma}\colon G \to G$ is an involution as well. You may use that the universal cover of a path-connected topological group is again a topological group.

(c) Prove that $G^{\tilde{\sigma}} = K \cong \mathbb{R}$.

(d) Prove that $\mathrm{Ad}_G(K) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2, \mathbb{R}))$.

(e) Show that $\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2, \mathbb{R})) \simeq \mathrm{SO}(2, \mathbb{R})/\{\pm 1\}$.

Exercise 5. (a) Let G be a connected topological group and $N \lhd G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

(b) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G. Show that $\mathrm{Ad}_G \colon G \to \mathrm{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \to \mathrm{Ad}_G(K) < \mathrm{GL}(\mathfrak{g}).$$