## Exercise Sheet 3

Exercise 1. Consider $G=\mathrm{SO}(1, n)^{\circ}$ with the involutive Lie group automorphism

$$
\sigma: G \rightarrow G, g \mapsto J_{n} g J_{n}
$$

where

$$
J_{n}=\left(\begin{array}{cc}
-1 & 0 \\
0 & I_{n}
\end{array}\right) \in \mathrm{SO}(1, n)
$$

Further let

$$
K=\left(\begin{array}{cc}
1 & 0 \\
0 & \mathrm{SO}(n)
\end{array}\right) \cong \mathrm{SO}(n)
$$

It can be shown that $(G, K, \sigma)$ is a Riemannian symmetric pair and that $G / K$ is isometric to $\mathbb{H}^{n}$.
(a) Show that $\Theta=d \sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ takes the form

$$
\Theta(X)=\left(\begin{array}{cc}
0 & -x^{t} \\
-x & D
\end{array}\right)
$$

for all

$$
X=\left(\begin{array}{cc}
0 & x^{t} \\
x & D
\end{array}\right) \in \mathfrak{g}=\mathfrak{s o}(1, n)
$$

Deduce that

$$
\begin{aligned}
& \mathfrak{p}=E_{-1}(\Theta)=\left\{\left(\begin{array}{cc}
0 & x^{t} \\
x & 0
\end{array}\right): x \in \mathbb{R}^{n}\right\} \\
& \mathfrak{k}=E_{1}(\Theta)=\left\{\left(\begin{array}{ll}
0 & 0 \\
0 & D
\end{array}\right): D \in \mathfrak{s o}(n)\right\} \cong \mathfrak{s o}(n) .
\end{aligned}
$$

(b) Let $\pi: G \rightarrow G / K$ denote the usual quotient map and set $\bar{X}:=d_{e} \pi(X) \in T_{o}(G / K)$ for all $X \in \mathfrak{g}$. Further let $\langle X, Y\rangle:=\frac{1}{2} \operatorname{tr}(X Y)$ for all $X, Y \in \mathfrak{p}$.
Show that

$$
R_{o}(\bar{X}, \bar{Y}) \bar{Z}=\langle X, Z\rangle \bar{Y}-\langle Y, Z\rangle \bar{X}
$$

for all $X, Y, Z \in \mathfrak{p}$. Deduce that $G / K$ has constant sectional curvature -1 .
Hint: You may use the following formula without proof:
The Riemann curvature tensor at $o \in M=G / K$ is given by

$$
R_{o}(\bar{X}, \bar{Y}) \bar{Z}=-\overline{[[X, Y], Z]}
$$

for all $\bar{X}, \bar{Y}, \bar{Z} \in T_{o} M$.
(c) Compute that

$$
\exp \left(t \cdot\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right)=\left(\begin{array}{ll}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{array}\right)
$$

for all $t \in \mathbb{R}$.
Exercise 2 (Closed adjoint subgroups of $\mathrm{SL}_{n}(\mathbb{R})$ and their symmetric spaces). Consider the Riemannian symmetric pair $(G, K, \sigma)$ where $G=\mathrm{SL}_{n}(\mathbb{R}), K=\mathrm{SO}(n, \mathbb{R})$ and $\sigma: \mathrm{SL}_{n}(\mathbb{R}) \rightarrow$ $\mathrm{SL}_{n}(\mathbb{R}), g \mapsto^{t}\left(g^{-1}\right)$. Further let $H \leq G$ be a closed, connected subgroup that is adjoint, i.e. it is closed under transposition $h \mapsto{ }^{t} h$.
(a) Show that $\left(H, H \cap K,\left.\sigma\right|_{H}\right)$ is again a Riemannian symmetric pair.
(b) Show that $i: H \hookrightarrow G$ descends to a smooth embedding $\varphi: H / H \cap K \hookrightarrow G / K$ such that its image is a totally geodesic submanifold of $G / K$.

Exercise 3 (The symplectic group $\operatorname{Sp}(2 n, \mathbb{R}))$. Let $H=\operatorname{Sp}(2 n, \mathbb{R})=\left\{g \in \mathrm{GL}_{2 n}(\mathbb{R}): g^{t} J g=J\right\}$ be the symplectic group, where

$$
J=\left(\begin{array}{cc}
0 & I_{n} \\
-I_{n} & 0
\end{array}\right)
$$

(a) Show that $\operatorname{Sp}(2 n, \mathbb{R}) \leq \operatorname{SL}(2 n, \mathbb{R})=$ : $G$ is a closed connected adjoint subgroup of $G$.

What can we deduce from exercise 2 about $\left(H, H \cap K,\left.\sigma\right|_{H}\right)$ ?
(b) Denote by $\omega: \mathbb{R}^{2 n} \times \mathbb{R}^{2 n} \rightarrow \mathbb{R}$ the standard symplectic form given by $\omega(x, y)=x^{t} J y$.

Show that $B: \mathbb{R}^{2 n} \times \mathbb{R}^{2 n} \rightarrow \mathbb{R},(x, y) \mapsto \omega(J x, y)$ is a symmetric positive definite bilinear form.
(c) An endomorphism $M \in \operatorname{End}\left(\mathbb{R}^{2 n}\right)$ is called a complex structure if $M^{2}=-\mathrm{Id}$. We say that $M$ is $\omega$-compatible if $(x, y) \mapsto \omega(M x, y)$ is a symmetric positive definite bilinear form. Denote the set of all $\omega$-compatible complex structures by $S_{2 n}$.
Show that $H=\operatorname{Sp}(2 n, \mathbb{R})$ acts on $S_{2 n}$ via conjugation and deduce that there is a bijection $S_{2 n} \cong H / H \cap K$.

