Exercise Sheet 4

Exercise 1 (Theorem II.27 - Decomposition of OSLA)**.** Let $(\mathfrak{g}, \theta)$ be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{i}$. We decomposed $\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{i}_+ \oplus \mathfrak{i}_-$ and defined $\mathfrak{u}_+ = [\mathfrak{i}_+, \mathfrak{i}_+]$ and $\mathfrak{u}_- = [\mathfrak{i}_-, \mathfrak{i}_-]$. $\mathfrak{u}_0$ is defined to be the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(a) Find an OSLA $(\mathfrak{g}, \theta)$, such that $\mathfrak{i}_0 = 0$, but $\mathfrak{u}_0 \neq 0$.

(b) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_\mathfrak{n} = B_\mathfrak{g}|_{\mathfrak{n} \times \mathfrak{n}}$.

(c) Find an example of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$, such that $B_\mathfrak{n} \neq B_\mathfrak{g}|_{\mathfrak{n} \times \mathfrak{n}}$.

(d) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Show that $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

This implies that $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$.

Hint: For connected G and $K < G$, there is an isomorphism

$$K/(K \cap Z(G)) \cong \operatorname{Ad}_G(K)$$

(compare Ex Sheet 2, exercise 5(b)). Use $\operatorname{Lie}(\operatorname{Ad}_G(K)) = \operatorname{ad}_{\operatorname{Lie}(G)}(\operatorname{Lie}(K))$.

Exercise 2 (Theorem II.33 - Decomposition of simply connected RSS)**.** (a) Let $H, N \lhd G$ be two normal subgroups. Show that $[N, H] \subset N \cap H$.

(b) Let $H, N < G$ be connected subgroups. Show that $[N, H]$ is a connected subgroup of G.

(c) Let M be a simply connected Riemannian symmetric space. Then $\mathfrak{g} = \operatorname{Lie}(\operatorname{Iso}(M)^\circ) = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We get corresponding Lie-subgroups G_0, G_+, G_- and their universal covers $\tilde{G}_0, \tilde{G}_+, \tilde{G}_-$. Let K_0, K_+, K_- be the Lie-subgroups associated to $\mathfrak{k}_0, \mathfrak{k}_+, \mathfrak{k}_-$, which come from the Cartan-decomposition of $\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-$.

Show that $(\tilde{G}_0, K_0), (\tilde{G}_+, K_+)$ and $(\tilde{G}_-, K_-)$ are Riemannian symmetric pairs.

Exercise 3 (Complexification and Killing form)**.** Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

(a) $K_0(X, Y) = K(X, Y)$ for all $X, Y \in \mathfrak{l}_0$;

(b) $K^{\mathbb{R}}(X, Y) = 2 \cdot \Re(K(X, Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.