Exercise Sheet 5

Exercise 1. Exhibit an explicit isomorphism between the two real Lie algebras $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$.

Hint: Consider the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2,\mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

Exercise 2 (Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$)**.** Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

Exercise 3 (CAT(0) spaces)**.** Let (X,d) be a complete CAT(0) space and $\emptyset \neq C \subseteq X$ be a convex closed subset of X. Prove that for every $x \in X$ there exists a unique point $p_C(x) \in C$ such that $d(x, p_C(x)) \leq d(x,y)$ for any $y \in C$.