# Solutions Exercise Sheet 1

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The Lie Groups lecture notes to which we refer in the solutions can be found on the course webpage.

1. Let $G = \mathrm{SL}_2(\mathbb{R})$. The aim of this exercise is to show that $G_\mathbb{Z} = \mathrm{SL}_2(\mathbb{Z})$ is a lattice in $G$.

   (a) Argue that $G_\mathbb{Z}$ is discrete in $G$ and that both $G$ and $G_\mathbb{Z}$ are unimodular.

**Solution:** Since $SL_n(\mathbb{Z})$ is contained in the topological space $\mathbb{Z}^{n\times n}$, a discrete set of $\mathbb{R}^{n\times n}$, the fact that $G_\mathbb{Z}$ is discrete in $G$ follows by restriction from $\mathbb{R}^{n\times n}$ to $G$. As a discrete group $G_\mathbb{Z}$ is unimodular (note that each counting measure on $G_\mathbb{Z}$ is a bi-invariant Haar measure). $G$ is unimodular as a closed normal Lie subgroup (Lie Groups Lecture Notes, Prop. 2.3) of the unimodular Lie group $GL_n(\mathbb{R})$. Recall that $GL_n(\mathbb{R})$ has bi-invariant Haar measure $|\det(x_{ij})|^{-1}dx_{11}\dots dx_{nn}$.

From this we know that $G/G_\mathbb{Z}$ admits a nonzero $G$-invariant measure $\mu$ which is unique up to a non-zero constant. In order to show that $G_\mathbb{Z}$ is a lattice we have to show that $\mu(G/G_\mathbb{Z}) < \infty$. For this, we will use the following fact:

   (b) Assume that there exists a measurable set $A \subseteq G$ of finite measure such that every $G_\mathbb{Z}$-orbit intersects $A$, i.e. for every $g \in G$ there exists some $\gamma \in G_\mathbb{Z}$ such that $g\gamma \in A$. Show that $\mu(G/G_\mathbb{Z})$ is finite.

**Solution:** Weil's Theorem (Lie groups Lecture Notes, Thm 2.4) for the (integrable) characteristic function $\chi_A$ of $A$ in $G$ states that

$$\mu(A) = \int_G \chi_A(g)dg = \int_{G/G_\mathbb{Z}} \left( \int_{G_\mathbb{Z}} \chi_A(\gamma h)dh \right) d(\gamma G_\mathbb{Z}).$$

By assumption, the inner integral is always greater than some absolute constant depending only on the Haar measure of $G_\mathbb{Z}$. (Recall that the Haar measure on $G_\mathbb{Z}$ must be a counting measure.) Thus, we infer from the above equation that $\mu(A) \geq c\mu(G/G_\mathbb{Z})$.

We delay the general proof for a moment to consider a classical case, namely $n = 2$. It is also closely related to symmetric spaces. In fact, the complex upper half plane $\mathcal{H}$ is a globally symmetric space, as we will see. As a Riemannian manifold it is isomorphic to the hyperbolic plane $H^2$, a symmetric space of non-compact type. It is even a complex manifold and the complex structure is compatible with its structure as a Riemannian manifold. Thus, it belongs to the important subclass of Hermitian symmetric spaces.

(c) Show that the map sending

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ to } z \longmapsto g \cdot z := \frac{az+b}{cz+d}$$

is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Bih}(\mathbb{H}^2)$, where $\mathrm{Bih}(\mathbb{H}^2)$ denotes the biholomorphic maps of the complex upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. Show that its kernel is $\{\pm I\}$ where $I$ denotes as usual the $2\times 2$ identity matrix.

**Solution:** Define the automorphy factor $j : SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ by

$$j(\gamma, z) = (cz+d), \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For all $z \in \mathcal{H}$ one has the trivial matrix relation

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}.$$

Given $\alpha, \beta \in SL_2(\mathbb{R})$ one now calculates $\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix}$ in two different ways: This yields both

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha\beta, z) \begin{pmatrix} (\alpha\beta)z \\ 1 \end{pmatrix}$$

and

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha, \beta z) j(\beta, z) \begin{pmatrix} \alpha(\beta z) \\ 1 \end{pmatrix}.$$

One deduces the automorphy relation

$$j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z) \text{ for all } \alpha, \beta \in SL_2(\mathbb{R}),\ z \in \mathcal{H},$$

and furthermore

$$(\alpha\beta)z = \alpha(\beta z) \text{ for } \alpha, \beta \in SL_2(\mathbb{R}),\ z \in \mathcal{H}.$$

This relation shows that the map $SL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$ is indeed a homomorphism. The remaining assertions are straightforward to verify.

(d) Prove that the induced homomorphism

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \{\pm I\} \to \mathrm{Bih}(\mathbb{H}^2)$$

of (c) is actually an isomorphism. For the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ from above determine the orbit $Gi$ and stabilizer $K$ of $i \in \mathbb{H}^2$. (Show also that $K$ is compact.) Using this, show that we have a diffeomorphism

$$G/K \longrightarrow \mathbb{H}^2,\ g \longmapsto g \cdot i.$$

**Solution:** It suffices to show that every biholomorphism of $\mathcal{H}$ is actually in the image of $SL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$. The Cayley transform

$$\varphi : z \longmapsto \frac{z-i}{z+i}$$

sends the upper half plane $\mathcal{H}$ biholomorphically onto the unit disc $\mathcal{D}$ around 0. Therefore, it establishes an isomorphism $\mathrm{Bih}(\mathcal{H}) \tilde{\to} \mathrm{Bih}(\mathcal{D})$ by $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$. Since

$$\left[\frac{1}{\sqrt{2i}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right] SL_2(\mathbb{R}) \left[\frac{1}{\sqrt{2i}} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\right] = SU_{1,1}(\mathbb{C})$$

with

$$SU_{1,1}(\mathbb{C}) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1\}$$

it suffices to show that every biholomorphism of $\mathcal{D}$ is in the image of the homomorphism $g \mapsto f_g$ defined as follows: For each

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU_{1,1}(\mathbb{C}),\ a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1,$$

the map

$$f_g : z \longmapsto (az+b)/(\bar{b}z + \bar{a})$$

is a biholomorphism of $\mathcal{D}$. Indeed,

$$\begin{aligned} |(az+b)/(\bar{b}z+\bar{a})| < 1 &\iff (az+b)(\bar{a}\bar{z}+\bar{b}) < (\bar{b}z+\bar{a})(b\bar{z}+a) \\ &\iff |a|^2|z|^2+|b|^2 < |b|^2|z|^2+|a|^2 \\ &\iff |z| < 1. \end{aligned}$$

shows that $f_g$ sends $\mathcal{D}$ to $\mathcal{D}$. It has inverse $f_{g^{-1}}$ because the argument from (3a) shows that $f_{g_2} \circ f_{g_1} = f_{g_2 g_1}$, i.e. $g \mapsto f_g$ is a homomorphism. Now, let $\varphi$ be an arbitrary biholomorphism of $\mathcal{D}$. Then,

$$\psi = (f_g \circ \varphi), \text{ where } g = \frac{1}{\sqrt{1+|\varphi(0)|^2}} \begin{pmatrix} 1 & -\varphi(0) \\ -\overline{\varphi(0)} & 1 \end{pmatrix} \in SU_{1,1}(\mathbb{C}),$$

is also a biholomorphism of $\mathcal{D}$ with the additional property that $\psi(0) = 0$. The classical Schwarz Lemma yields $\psi(z) = e^{i\theta}z$, $\theta \in [0, 2\pi)$. One infers easily $\varphi \in SU_{1,1}(\mathbb{C})$ from this and that $PSL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$ is an isomorphism.

Since

$$\begin{pmatrix} y^{1/2} & x \\ 0 & y^{-1/2} \end{pmatrix} i = x + iy$$

for any $x \in \mathbb{R}$, $y \in \mathbb{R}^+$ the orbit $Gi$ equals $\mathcal{H}$. Furthermore, given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ the following equivalences hold

$$gi = i \iff \frac{ai+b}{ci+d} = i \iff ai + b = -c + id.$$

Taking real and imaginary parts, one deduces that $SO_2(\mathbb{R})$ is the stabilizer of $i$. That $G/K \longrightarrow \mathcal{H}$ is a diffeomorphism follows from standard arguments of differential geometry: In fact, it is a homeomorphism by Helgason, Theorem II.3.2, and a diffeomorphism by Helgason, Theorem II.4.3.(a).

(e) Set $K = \mathrm{SO}_2(\mathbb{R})$,

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R},\, a > 0 \right\},$$

$$A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \mid y \in \mathbb{R}^+ \right\}, \text{ and}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P \times K \longrightarrow G,\ (p, k) \longmapsto pk$$

and

$$N \times A \longrightarrow P,\ (n, a) \longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms?
Show that $P$ is a semidirect product $N \rtimes A$ and that we have the diffeomorphism $N \times A \cong \mathbb{H}^2$.

**Solution:** Both $P \cap K = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ and $N \cap A = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ are easily verifiable identities. The two maps $P \times K \to G$ and $N \times A \to P$ are hence injective. Let us show next that $P \times K \to G$ is surjective: Given $g \in G$ we can regard the matrix of $g = (f_1, f_2)$, $f_i \in \mathbb{R}^2$, as determining a basis $\{f_1, f_2\}$ of $\mathbb{R}^2$. We introduce on $\mathbb{R}^2$ the canonical scalar product $\langle \cdot, \cdot \rangle$. Now, we can use the Gram-Schmidt algorithm for $\langle \cdot, \cdot \rangle$ on $\{f_1, f_2\}$ to find a matrix $p \in P$

such that $p^{-1}g \in K$. In addition, a simple algebraic argument shows that $N \times A \to P$ is surjective. Neither $P \times K \to G$ nor $N \times A \to P$ is a homomorphism. However, since $NA = P$, $N \cap A = \{1\}$ and $N \lhd P$ one has $P = N \rtimes A$.

*Caveat: Showing that a differentiable map is bijective does not suffice to prove that it is a diffeomorphism, i.e. has a differentiable inverse. However, the Gram-Schmidt algorithm provides us directly with a differentiable inverse.*

Finally, $\mathcal{H} \cong G/K \cong (N \times A \times K)/K \cong N \times A$.

(f) Prove that $K$ is unimodular by showing that $d\mu_{\mathbb{H}^2} = y^{-2}dxdy$, $z = x+iy$, is a $G$-invariant volume form on the $G$-homogeneous space $\mathbb{H}^2$.

**Solution**: The derivative of $f_g : z \mapsto \frac{az+b}{cz+d}$ is

$$\frac{a(cz+d) - c(az+b)}{(cz+d)^2} = (cz+d)^{-2}.$$

The Cauchy-Riemann differential equations imply

$$\begin{pmatrix} \frac{\partial(\operatorname{Re} f_g)}{\partial x} & \frac{\partial(\operatorname{Re} f_g)}{\partial y} \\ \frac{\partial(\operatorname{Im} f_g)}{\partial x} & \frac{\partial(\operatorname{Im} f_g)}{\partial y} \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(cz+d)^{-2} & -\operatorname{Im}(cz+d)^{-2} \\ \operatorname{Im}(cz+d)^{-2} & \operatorname{Re}(cz+d)^{-2} \end{pmatrix}$$

and the determinant $\Delta$ of this matrix is

$$\left[\operatorname{Re}(cz+d)^{-2}\right]^2 + \left[\operatorname{Im}(cz+d)^{-2}\right]^2 = |cz+d|^{-4}.$$

From this we deduce

$$g^*(dxdy) = \Delta dxdy = |cz+d|^{-4}dxdy.$$

Furthermore,

$$g^*(y^{-2}) = \operatorname{Im}(\frac{az+b}{cz+d})^{-2} = y^{-2}|cz+d|^4$$

and therefore

$$g^*(d\mu_{\mathcal{H}}) = g^*(y^{-2}dxdy) = y^{-2}dxdy = d\mu_{\mathcal{H}}.$$

Now, the assertion follows from Weil's theorem

(g) Let

$$\mathcal{F} := \{z \in \mathbb{H}^2 \mid (|z| > 1 \text{ and } -1/2 \leq \operatorname{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \leq \operatorname{Re}(z) \leq 0)\}.$$

Show that for all $z \in \mathbb{H}^2$ the orbit $G_{\mathbb{Z}}z$ intersects $\mathcal{F}$ in a unique point.
Hint: For every $G_{\mathbb{Z}}$-orbit $G_{\mathbb{Z}}z$, $z \in \mathbb{H}^2$, consider $w \in G_{\mathbb{Z}}z$ with maximal imaginary part.

**Solution**: First of all, note that

$$\operatorname{Im}(\frac{az+b}{cz+d}) = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

If $c = 0$ then $\operatorname{Im}(z)/|cz+d|^2 = \operatorname{Im}(z)/|d|^2$ and $\operatorname{Im}(z)/|cz+d|^2 \leq \operatorname{Im}(z)^{-1}|c|^2$ elsewise. Hence, given some $z \in \mathcal{H}$ the function

$$G_{\mathbb{Z}} \longrightarrow \mathcal{H},\ \gamma \mapsto \operatorname{Im}(\gamma z)$$

obtains a maximum on $G_{\mathbb{Z}}$, i.e. there exists $\gamma_0 \in G_{\mathbb{Z}}$ such that

$$\operatorname{Im}(\gamma_0 z) = \max_{\gamma \in G_{\mathbb{Z}}}\{\operatorname{Im}(\gamma z)\}.$$

Write $w = \gamma_0 z \in G_{\mathbb{Z}} z$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts as a translation $z \mapsto (z+1)$ on $\mathcal{H}$ we may assume that $-1/2 \leq \mathrm{Re}(w) < 1/2$. In addition, since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts as inversion $z \mapsto -z^{-1}$ on $\mathcal{H}$

and

$$\mathrm{Im}(-\frac{1}{w}) = (-\frac{1}{w} + \frac{1}{\bar{w}})/(2i) = \frac{\mathrm{Im}(w)}{|w|^2}$$

one clearly has $|w| \geq 1$. It remains to show that we may impose $-1/2 \leq \mathrm{Re}(w) \leq 0$ if $|w| = 1$ but this follows easily from the fact that $z \mapsto -z^{-1}$ sends

$$\{z \in \mathcal{H} \mid (|z| = 1 \text{ and } 0 < \mathrm{Re}(z) \leq 1/2)\}$$

to

$$\{z \in \mathcal{H} \mid (|z| = 1 \text{ and } -1/2 \leq \mathrm{Re}(z) < 0)\}.$$

Thus, each orbit $G_{\mathbb{Z}} z$ intersects $\mathcal{F}$. To actually show that $\mathcal{F}$ is a fundamental domain it remains to prove that $\{z, \gamma z\} \subset \mathcal{F}$ for some $\gamma \in G_{\mathbb{Z}}$ implies $\gamma z = z$, i.e. $\gamma \in Stab(z)$. This can be done using considerations similar to those above and we leave these to the reader.

(h) Show that the volume of $\mathcal{F}$ with respect to $\mu_{\mathbb{H}^2}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

**Solution**: This is a simple calculation:

$$\begin{aligned}
\int_{\mathcal{F}} d\mu_{\mathcal{H}} &= \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} y^{-2} dy dx \\
&= \int_{-1/2}^{1/2} \left[-y^{-1}\right]_{y=\sqrt{1-x^2}}^{y=\infty} dx \\
&= \int_{-1/2}^{1/2} (1-x^2)^{-1/2} dx \\
&= [\arcsin(x)]_{x=-1/2}^{x=1/2} = \pi/3.
\end{aligned}$$

The second assertion follows from part (b) applied to $A = \mathcal{F}$.

2. Consider the hyperbolic $n$-space

$$\mathbb{H}^n = \left\{p \in \mathbb{R}^{n+1} \colon b(p,p) = -1 \text{ and } p_{n+1} \geq 1\right\}$$

defined by the bilinear form $b(p,q) = p_1 q_1 + \ldots + p_n q_n - p_{n+1} q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \colon \begin{array}{c} \text{There exists a smooth path } \gamma \colon (-1,1) \to \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot\gamma(0) = x \end{array} \right\}.$$

(a) Show that $T_p\mathbb{H}^n = \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$.

**Solution.** Let $x \in T_p\mathbb{H}^n$. Let $\gamma \colon (-1,1) \to \mathbb{H}^n$ be a smooth path such that $\gamma(0) = p$ and $\dot\gamma(0) = x$. For every $t \in (-1,1)$, $b(\gamma(t), \gamma(t)) = -1$, since $\gamma$ takes values in $\mathbb{H}^n$. We write $\gamma(t) = (\gamma_1(t), \cdots, \gamma_{n+1}(t))$. Taking derivatives results in

$$0 = \frac{d}{dt} b(\gamma(t), \gamma(t)) = \frac{d}{dt}\left(\sum_{i=1}^n \gamma_i(t)^2 - \gamma_{n+1}^2\right) = \sum_{i=1}^n 2\gamma_i(t)\dot\gamma_i(t) - 2\gamma_{n+1}(t)\dot\gamma_{n+1}(t)$$

and at $t = 0$ this is

$$0 = \sum_{i=1}^n \gamma_i(0)\dot\gamma_i(0) - \gamma_{n+1}(0)\dot\gamma_{n+1}(0) = \sum_{i=1}^n p_i x_i - p_{n+1} \cdot x_{n+1} = b(p,x).$$

We have shown that $T_p\mathbb{H}^n \subset \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$ but since $\dim T_p\mathbb{H}^n = n$ we have equality.

(b) Show that $g_p = b|_{T_p\mathbb{H}^n} \colon T_p\mathbb{H}^n \times T_p\mathbb{H}^n \to \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p\mathbb{H}^n$. This means that $g_p$ is a scalar product and $(\mathbb{H}^n, g)$ is a Riemannian manifold. *Hint*: Use (a) and the Cauchy-Schwarz-inequality on $\mathbb{R}^n$.

**Solution.** Bilinearity and symmetry $b(x,y) = b(y,x)$ follow directly. To show positive definiteness, we use the definition of $\mathbb{H}^n$ and (a) to write

$$p = \left(\overrightarrow{p}, \sqrt{|\overrightarrow{p}|^2 + 1}\right) \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

$$x = \left(\overrightarrow{x}, \frac{< \overrightarrow{p}, \overrightarrow{x} >}{\sqrt{|\overrightarrow{p}|^2 + 1}}\right) \in T_p\mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

where $< \cdot\, , \cdot >$ is the standard scalar product in $\mathbb{R}^n$. To show positive definiteness it remains to prove that for all $x \in T_p\mathbb{H}^n$

$$b(x,x) \geq 0.$$

Indeed, by the Cauchy-Schwarz-inequality

$$< p, x >^2 \leq |\overrightarrow{p}|^2 |\overrightarrow{x}|^2 \leq |\overrightarrow{p}|^2 |\overrightarrow{x}|^2 + |\overrightarrow{x}|^2 = (|\overrightarrow{p}|^2 + 1)|\overrightarrow{x}|^2$$

and thus

$$|\overrightarrow{x}|^2 \geq \frac{< p, x >^2}{|\overrightarrow{p}|^2 + 1}$$

and

$$b(x,x) = |\overrightarrow{x}|^2 - \frac{< p, x >^2}{|\overrightarrow{p}|^2 + 1} \geq 0.$$

(c) Show that the map $s_p\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, q \mapsto -2p \cdot b(p,q) - q$ defines a well defined geodesic symmetry of $\mathbb{H}^n$, i.e. it is an involution, with an isolated fixed point $p$. This means that the hyperbolic plane $\mathbb{H}^n$ is a symmetric space.

**Solution.** To see that $s_p$ is well-defined we write

$$p = \begin{pmatrix} \overrightarrow{p} \\ \sqrt{|\overrightarrow{p}|^2+1} \end{pmatrix}, \quad q = \begin{pmatrix} \overrightarrow{q} \\ \sqrt{|\overrightarrow{q}|^2+1} \end{pmatrix} \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}.$$

We have

$$s_p(q) = -2pb(p,q) - q = \begin{pmatrix} -2\overrightarrow{p}b(p,q) - \overrightarrow{q} \\ -2\sqrt{|\overrightarrow{p}|^2+1}b(p,q) - \sqrt{|\overrightarrow{q}|^2+1} \end{pmatrix}$$

where

$$b(p,q) = < \overrightarrow{p}, \overrightarrow{q} > - \sqrt{|\overrightarrow{p}|^2+1}\sqrt{|\overrightarrow{q}|^2+1}.$$

The calculation

$$\begin{aligned} b(s_p(q), s_p(q)) &= 4|\overrightarrow{p}|^2 b(p,q)^2 + 4 < \overrightarrow{p}, \overrightarrow{q} > b(p,q) + |\overrightarrow{q}|^2 \\ &\quad - \left[ 4(|\overrightarrow{p}|^2+1)b(p,q)^2 + 4\sqrt{|\overrightarrow{p}|^2+1}\sqrt{|\overrightarrow{q}|^2+1}b(p,q) + |\overrightarrow{q}|^2 + 1 \right] \\ &= 4 < \overrightarrow{p}, \overrightarrow{q} > b(p,q) - 4b(p,q)^2 - 4\sqrt{|\overrightarrow{p}|^2+1}\sqrt{|\overrightarrow{q}|^2+1}b(p,q) - 1 \\ &= 4b(p,q)b(p,q) - 4b(p,q)^2 - 1 = -1 \end{aligned}$$

shows that $s_p(q) \in \mathbb{H}^2$.

Note that $s_p(p) = -2p(-1) - p = p$ is a fixed point.

Next we show that $s_p$ is an isometry. We need to look at the differential

$$d_p s_p\colon T_pM \to T_{s_p(p)}M = T_pM.$$

of $s_p\colon q \mapsto -2pb(p,q) - q$. If we write the points $q, p \in \mathbb{H}^n \subset \mathbb{R}^{n+1}$ in the standard basis $\{e_i\}_i$, we get the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x_i} b(p,\cdot) &= \begin{cases} p_i & \text{if } i \leq n \\ -p_{n+1} & \text{if } i = n+1 \end{cases} \\ \frac{\partial}{\partial x_i} s_p &= \begin{cases} -2p \cdot p_i - e_i & \text{if } i \leq n \\ 2p \cdot p_{n+1} - e_{n+1} & \text{if } i = n+1 \end{cases} \end{aligned}$$

and thus for $v \in T_pM$ we have

$$
\begin{aligned}
(d_p s_p)v &= \begin{pmatrix} -2p_1^2 - 1 & -2p_1p_2 & \cdots & 2p_1p_{n+1} \\ -2p_2p_1 & -2p_2^2 - 1 & \cdots & 2p_2p_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ -2p_{n+1}p_1 & -2p_{n+1}p_2 & \cdots & 2p_{n+1}^2 - 1 \end{pmatrix} v \\
&= \begin{pmatrix} -2p_1^2v_1 - 2p_1p_2v_2 - \cdots + 2p_1p_{n+1}v_{n+1} \\ -2p_2p_1v_1 - 2p_2^2v_2 - \cdots + 2p_2p_{n+1}v_{n+1} \\ \vdots \\ -2p_{n+1}p_1v_1 - 2p_{n+1}p_2v_2 - \cdots + 2p_{n+1}^2v_{n+1} \end{pmatrix} - v \\
&= \begin{pmatrix} -2b(p,v)p_1 \\ -2b(p,v)p_2 \\ \vdots \\ -2b(p,v)p_n \end{pmatrix} - v = -v
\end{aligned}
$$

where we used that $b(p, v) = 0$ from part (a). By bilinearity from (b)

$$g_{s_p(p)}((D_ps_p)v, (D_ps_p)w) = g_p(-v, -w) = g_p(v, w),$$

so $s_p$ is an isometry.

We need to show that $s_p$ is a symmetry. That $p$ is an isolated fixed point of $s_p$ can be seen by the following argument. Let $q \in \mathbb{H}^n$ be a fixed point $s_p(q) = q$, then $-2pb(p, q) - q = q$, so $q = -b(p, q)p$, in particular $q = \lambda p$ is a scaled version of $p$ for $\lambda = -b(p, q)$. But since $p, q \in \mathbb{H}^n$, $-1 = b(q, q) = b(\lambda p, \lambda p) = \lambda^2 b(p, p) = -1$, so $\lambda = \pm 1$. The $\lambda = -1$ solution corresponds to $q_{n+1} < 0$ which is excluded since $\mathbb{H}^n$ is only the upper hyperboloid. We showed that $q = p$ is the only fixed point of $s_p$, in particular it is an isolated fixed point.

By lemma II.5 of the lecture, $d_ps_p = -\operatorname{Id}_{T_p\mathbb{H}^n}$ is equivalent to $s_p \circ s_p = \operatorname{Id}_{\mathbb{H}^n}$. Alternatively the calculation

$$
\begin{aligned}
s_p \circ s_p(q) &= s_p(-2pb(p, q) - q) \\
&= -2pb(p, -2pb(p, q) - q) - (-2pb(p, q) - q) \\
&= 4pb(p, q)b(p, p) + 2pb(p, q) + 2pb(p, q) + q = q
\end{aligned}
$$

shows the same.

This concludes the proof, that $\mathbb{H}^n$ is a symmetric space.

3. Show that $A \mapsto gAg^t$ defines a group action of $\mathrm{SL}(n, \mathbb{R}) \ni g$ on

$$\mathcal{P}(n) = \left\{ A \in M_{n\times n}(\mathbb{R})\colon A = A^t,\ \det A = 1,\quad A >> 0 \right\}.$$

Show that this action is transitive, i.e. $\forall A, B \in \mathcal{P}(n)\ \exists g \in \mathrm{SL}(n, \mathbb{R})\colon gAg^t = B$. You may use the Linear-Algebra-fact that symmetric matrices are orthogonally diagonalizable, i.e. if $A = A^t$, then $\exists Q \in \mathrm{SO}(n, \mathbb{R})$ such that $QAQ^t$ is diagonal.

**Solution.** We write the group action as $g.A = gA\,{}^tg$. We first need to show that the action is well defined.

Symmetry: ${}^t(g.A) = {}^t(gA\,{}^tg) = g\,{}^tA\,{}^tg = gA\,{}^tg = g.A$.

Determinant: $\det(g.A) = \det g \det A \det{}^tg = \det A = 1$.

Positive definiteness: Let $x \in \mathbb{R}^n \setminus 0$. ${}^txg.Ax = {}^txgA\,{}^tgx = {}^t({}^tgx)A\,{}^tgx > 0$, since ${}^tgx \in \mathbb{R}^n \setminus 0$.

Next, we check the two axioms of a group action.

Identity: $\mathrm{Id}_{\mathrm{SL}(n,\mathbb{R})}.A = \mathrm{Id}\,A\,\mathrm{Id} = A$.

Compatibility: $(gh).A = ghA\,{}^t(gh) = ghA\,{}^th\,{}^tg = g(h.A)\,{}^tg = g.(h.A)$.

It remains to show that the action is transitive. Let $A, B \in \mathcal{P}(n)$. We can use linear algebra to get $Q, R \in \mathrm{SO}(n) < \mathrm{SL}(n,\mathbb{R})$ such that $Q.A$ and $R.B$ are diagonal, have determinant 1 and are positive definite (by the well-definedness of the group action). Positive definiteness implies that all entries are non-negative. Then the matrix $\Lambda = (Q.A) \cdot (R.B)^{-1}$ is also diagonal, has determinant 1 and positive elements on the diagonal. We can therefore take the component wise root $\sqrt{\Lambda}$ of $\Lambda$. Define $g = Q^{-1}\sqrt{\Lambda}R \in \mathrm{SL}(n,\mathbb{R})$ and use the fact that $R.B$ commutes with $\sqrt{\Lambda}$ since they are diagonal to see that

$$\begin{aligned} g.B &= Q^{-1}.\sqrt{\Lambda}.R.B = Q^{-1}.\sqrt{\Lambda}(R.B)\,{}^t\sqrt{\Lambda} = Q^{-1}.\left(\sqrt{\Lambda}\,{}^t\sqrt{\Lambda} \cdot R.B\right) \\ &= Q^{-1}.(\Lambda \cdot R.B) = Q^{-1}.((Q.A)(R.B)^{-1}(R.B)) = Q^{-1}.Q.A = A. \end{aligned}$$

this shows that from any point $B \in \mathcal{P}(n)$ you can go to any point $A \in \mathcal{P}(n)$ by the action of $\mathrm{SL}(n,\mathbb{R})$, i.e. the action is transitive.