Solutions Exercise Sheet 2

Exercise 1 (Compact Lie groups as symmetric spaces)**.** Let G be a compact connected Lie group and let

$$G^* = \{(g, g) \in G \times G : g \in G\} < G \times G$$

denote the diagonal subgroup.

(a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

Solution. Consider the mapping $\sigma\colon (g_1, g_2) \mapsto (g_2, g_1)$. This is an involutive automorphism of the product group $G \times G$. The fixed set of σ is precisely the diagonal G^*. It follows that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair. The coset space $G \times G/G^*$ is *diffeomorphic* to G under the mapping

$$\begin{aligned} \varphi : G \times G/G^* &\to G \\ [(g_1, g_2)] &\mapsto g_1 g_2^{-1}. \end{aligned}$$

(b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

Solution. By Theorem II.16 from the lecture, $G \times G/G^*$ is a Riemannian globally symmetric space with respect to any $G \times G$-invariant metric (and there is one). Notice that if a Riemannian metric on $G \times G/G^*$ is $G \times G$-invariant if and only if the corresponding Riemannian metric on G is bi-invariant. This follows from the identity $\varphi \circ L_{g_1, g_2} = R_{g_2^{-1}} \circ L_{g_1} \circ \varphi$ for every $g_1, g_2 \in G$. Thus G is a Riemannian symmetric space with respect to any bi-invariant metric (and there is one). (We remark that not every Lie group admits a bi-invariant metric, but compact Lie groups do).

(c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the *Riemannian* exponential map from $\mathfrak{g}$ into the Riemannian globally symmetric space G.

Solution. Note that the product algebra $\mathfrak{g} \times \mathfrak{g}$ is the Lie algebra of $G \times G$. Let

$\exp_G : \mathfrak{g} \to G$ be the exponential map of G

$\exp_{G\times G} = \exp_G \times \exp_G : \mathfrak{g} \times \mathfrak{g} \to G \times G$ be the exponential map of $G \times G$

$\operatorname{Exp}_e : \mathfrak{g} \cong T_e G \to G$ be the Riemannian exponential map of G.

We want to show that $\exp_G X = \operatorname{Exp}_e X$ for all $X \in \mathfrak{g}$.

Let $\pi : G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$ be the projection. Then by Theorem II.21 of the lecture applied to $G \times G$ we have

$$\pi \circ \exp_{G\times G}|_{\mathfrak{p}} = \operatorname{Exp}_e \circ d_{e,e}\pi|_{\mathfrak{p}},$$

where $\mathfrak{p} = E_{-1}(d_{e,e}\sigma) \subset \mathfrak{g} \times \mathfrak{g}$.

Let now $X \in \mathfrak{g}$. Then $(X, -X) \in \mathfrak{p}$ and therefore

$$\pi(\exp_{G\times G}(X, -X) = \operatorname{Exp}_e(d_{e,e}\pi(X, -X)).$$

Since $d_{e,e}\pi(X, Y) = X - Y$ and $\exp_{G\times G}(X, -X) = (\exp_G(X), \exp_G(-X))$ we have

$$\exp(2X) = \exp_G(X)\exp_G(-X)^{-1} = \operatorname{Exp}_e(2X),$$

where we used that the Lie group exponential is a one-parameter subgroup. Since X was arbitrary, this concludes the proof.

Exercise 2 (Compact semisimple Lie groups as symmetric spaces)**.** A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Solution. Let π and σ be as in the above solution. The map $d\pi$ maps the -1 eigenspace of $d\sigma$ onto $\mathfrak{g}$ as follows: $d\pi(X, -X) = 2X$. Using this, we can check that

$$2B_{\mathfrak{g}\times\mathfrak{g}}((X, -X), (X, -X)) = B_{\mathfrak{g}}(2X, 2X),$$

which is equivalent to $Q = 2Q^*$.

Exercise 3 (Closed differential forms)**.** Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\operatorname{Isom}(M)^\circ$. Prove that $d\omega = 0$.

Solution. Let s_m denote the geodesic symmetry at some point $m \in M$, and let $\omega \in \Omega^p(M)$ be an invariant differential p-form on M. Because $d_m s_m = -\operatorname{Id}\colon T_pM \to T_pM$, we get $(s_m^*\omega)_m = (-1)^p \omega_m$ at the point $m \in M$. Because ω is invariant, $s_m^*\omega$ is invariant as well. Because $\operatorname{Iso}(M)^\circ$ acts transitively, invariant differential forms are determined by their value at a single point such that

$$s_m^*\omega = (-1)^p\omega$$

on all of M.

Therefore, we obtain

$$d\omega = (-1)^p d(s_m^*\omega) = (-1)^p s_m^* d\omega = (-1)^{2p+1} d\omega,$$

whence $d\omega = 0$.

Exercise 4 (A symmetric space with non-compact K)**.** Let $G = \widetilde{\operatorname{SL}(2, \mathbb{R})}$ and $K = \widetilde{\operatorname{SO}(2, \mathbb{R})}$. The aim of this exercise is to show that (G, K) is a symmetric pair with non-compact K.

(a) Prove that $\sigma\colon \operatorname{SL}(2, \mathbb{R}) \to \operatorname{SL}(2, \mathbb{R}), g \mapsto {}^t g^{-1}$ is an involution.

Solution. Note that σ needs to be an automorphism. Being a homomorphism $\sigma(gh) = \sigma(g)\sigma(h)$ and $\sigma \circ \sigma = \mathrm{Id}$ follows directly from properties of the inverse and the transpose. Bijectivity follows from $\sigma \circ \sigma = \mathrm{Id}$. Finally, most matrices in $\mathrm{SL}(2, \mathbb{R})$ are not fixed by σ, so $\sigma \neq \mathrm{Id}$.

(b) By covering space theory we can lift σ to the universal cover G. Prove that $\tilde{\sigma}\colon G \to G$ is an involution as well. You may use that the universal cover of a path-connected topological group is again a topological group.

Solution. Recall from covering space theory the following fact:

Let $\pi\colon C \to X$ be a cover and $f\colon Y \to X$ a continuous map. Pick $y_0 \in Y$ and $c_0 \in C$, which lies over $f(y_0)$, i.e. $\pi(c_0) = f(y_0)$. If Y is simply connected, then there exists a unique lift $\tilde{f}\colon Y \to C$ with $\pi \circ \tilde{f} = f$ and $\tilde{f}(y_0) = c_0$. In our case, $Y = C = G$ is the universal cover and thus simply connected. Let us write $\pi\colon G \to \mathrm{SL}(2, \mathbb{R})$, and $f = \sigma \circ \pi$. Fix an element $\tilde{\mathrm{Id}}$ in the universal cover with $\pi(\tilde{\mathrm{Id}}) = \mathrm{Id}$, then we get a unique map $\tilde{\sigma}\colon G \to G$ with $\tilde{\sigma}(\tilde{\mathrm{Id}}) = \tilde{\mathrm{Id}}$ (Note: $\tilde{\sigma}$ is called the lift of σ, even though strictly speaking it is the lift of $\sigma \circ \pi$).

We have to show that $\tilde{\sigma}$ is a homomorphism: For this, consider the map

$$\begin{aligned} h\colon G \times G &\to G \\ (g,h) &\mapsto \tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h) \end{aligned}$$

Since $\pi(gh) = \pi(g)\pi(h)$ (the multiplication in the universal covering is the lift of the multiplication in the group), π is a homomorphism. We have

$$\begin{aligned} \pi(h(g,h)) &= \pi(\tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h)) \\ &= \pi(\tilde{\sigma}(gh)^{-1})\pi(\tilde{\sigma}(h))\pi(\tilde{\sigma}(h)) \\ &= \pi(\tilde{\sigma}(gh))^{-1}\pi(\tilde{\sigma}(h))\pi(\tilde{\sigma}(h)) \\ &= \sigma(\pi(gh))^{-1}\sigma(\pi(g))\sigma(\pi(h)) \\ &= \sigma(\pi(g)\pi(h))^{-1}\sigma(\pi(g)\pi(h)) \\ &= \mathrm{Id} = \pi(\tilde{\mathrm{Id}}) \end{aligned}$$

so h is a lift of $\pi \circ h$ and so is the constant function $(g,h) \mapsto \tilde{\mathrm{Id}}$. Since the lift is unique we have $h(g,h) = \tilde{\mathrm{Id}}$, i.e. $\tilde{\sigma}(gh) = \tilde{\sigma}(g)\tilde{\sigma}(h)$.

The composition $\tilde{\sigma} \circ \tilde{\sigma}$ satisfies $\pi \circ \tilde{\sigma} \circ \tilde{\sigma} = \sigma \circ \pi \circ \tilde{\sigma} = \sigma \circ \sigma \circ \pi = \pi$, so $\tilde{\sigma} \circ \tilde{\sigma}$ as well as the constant function $g \mapsto \tilde{\mathrm{Id}}$ is a lift of π. By the uniqueness, we get that $\tilde{\sigma} \circ \tilde{\sigma}(g) = \tilde{\mathrm{Id}}$ for all $g \in G$. In particular, $\tilde{\sigma}$ is an automorphism.

Finally, since σ is not the identity, its lift is also not the lift of the identity, i.e. $\tilde{\sigma}$ is not the identity-map on G. This concludes the proof that $\tilde{\sigma}$ is an involution.

(c) Prove that $G^{\tilde{\sigma}} = K \cong \mathbb{R}$.

Solution. The map $\sigma|_{\mathrm{SO}(2)}\colon \mathrm{SO}(2) \to \mathrm{SO}(2)$ is the identity. Its lift $\tilde{\sigma}|_{\widetilde{\mathrm{SO}(2)}}\colon \widetilde{\mathrm{SO}(2)} \to \widetilde{\mathrm{SO}(2)}$ therefore also has to be the identity by uniqueness of the lift. So if $g \in K = \widetilde{\mathrm{SO}(2)}$, then $\tilde{\sigma}(g) = g$, i.e. $g \in G^{\tilde{\sigma}}$.

If on the other hand $g \in G$ satisfies $\tilde{\sigma}(g) = g$, then $\pi(g) = \pi(\tilde{\sigma}(g)) = \sigma(\pi(g))$, so $\pi(g) \in \mathrm{SL}(2,\mathbb{R})^{\sigma} = \mathrm{SO}(2)$. Thus $g \in \widetilde{\mathrm{SO}(2)}$.

This implies $G^{\tilde{\sigma}} = K$.

(d) Prove that $\mathrm{Ad}_G(K) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R}))$.

Solution. The Lie algebra $\mathfrak{g}$ only depends on a neighborhood, so

$$\mathrm{Lie}(\mathrm{SL}(2,\mathbb{R})) = \mathfrak{g} = \mathrm{Lie}\left(\widetilde{\mathrm{SL}(2,\mathbb{R})}\right).$$

Since the left-multiplication on the universal cover is the lift of the left-multiplication of $\mathrm{SL}(2,\mathbb{R})$, they can be identified in a small neighborhood around $o = \mathrm{Id}$. The adjoint representation $\mathrm{Ad}(g) = d_o \mathrm{Int}(g)$ is a derivative at a point and thus also only depends on a neighborhood. We conclude that image of the adjoint representations is equal.

(e) Show that $\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) \simeq \mathrm{SO}(2,\mathbb{R})/\{\pm 1\}$.

Solution. The elements g in the kernel satisfy $X = gXg^{-1}$ for all $X \in \mathfrak{sl}(2,\mathbb{R}) = \left\{X \in \mathbb{R}^{2\times 2} \colon \mathrm{tr}(X) = 0\right\}$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

then we have $X = gXg^{-1}$ implies

$$Xg = \begin{pmatrix} ax + bz & ay - bx \\ cx + dz & cy - dx \end{pmatrix} = \begin{pmatrix} ax + cy & bx + dy \\ az - cx & bz - dx \end{pmatrix} = gX$$

so $bz = cy$ for all $z, y \in \mathbb{R}$, so $b = 0 = c$. So we have $ay = dy$ and $dz = az$ which imply $a = d$. Since $g \in \mathrm{SO}(2)$, $\det(g) = ad = a^2 = 1$. So $a = \pm 1$. We conclude that g has to be $\pm\mathrm{Id}$. And indeed both $\pm\mathrm{Id}$ are in $\mathrm{SO}(2)$. By the isomorphism-theorem we have

$$\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) \cong \mathrm{SO}(2,\mathbb{R})/\pm\mathrm{Id}.$$

Exercise 5. (a) Let G be a connected topological group and $N \lhd G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

(b) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G. Show that $\mathrm{Ad}_G \colon G \to \mathrm{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \to \mathrm{Ad}_G(K) < \mathrm{GL}(\mathfrak{g}).$$