# Solution Exercise Sheet 4

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**Exercise 1** (Theorem II.27 - Decomposition of OSLA)**.** Let $(\mathfrak{g}, \theta)$ be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{i}$. We decomposed $\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{i}_+ \oplus \mathfrak{i}_-$ and defined $\mathfrak{u}_+ = [\mathfrak{i}_+, \mathfrak{i}_+]$ and $\mathfrak{u}_- = [\mathfrak{i}_-, \mathfrak{i}_-]$. $\mathfrak{u}_0$ is defined to be the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(a) Find an OSLA $(\mathfrak{g}, \theta)$, such that $\mathfrak{i}_0 = 0$, but $\mathfrak{u}_0 \neq 0$.

**Solution.** The idea is to have a large $\mathfrak{u}$ and a small $\mathfrak{i}$. This means that $\theta$ should fix lots of points. For example one can take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(3)$ and define $\theta = \theta_{\mathfrak{sl}(2,\mathbb{R})} \times \mathrm{Id}_{\mathfrak{so}(3)}$, where $\theta_{\mathfrak{sl}(2,\mathbb{R})} = \mathrm{D}_e\,\sigma$ (for $\sigma(g) = {}^tg^{-1}$) is the usual Cartan-involuion on $\mathfrak{sl}(2, \mathbb{R})$. Then $\mathfrak{u} = E_1\theta = \mathfrak{k} \times \mathfrak{so}(3)$ and $\mathfrak{i} = E_{-1}\theta = \mathfrak{p} \times 0$, where $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan-decomposition of $\mathfrak{sl}(2, \mathbb{R})$.

We need to check that $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie-algebra (OSLA): $\theta$ is an involutive automorphism since $\theta_{\mathfrak{sl}(2,\mathbb{R})}$ and $\mathrm{Id}_{\mathfrak{so}(3)}$ are and we also have $\theta \neq \mathrm{Id}_{\mathfrak{g}}$. The definition of OSLA requires $\mathfrak{u}$ to be compactly-embedded in $\mathfrak{g}$, i.e. $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ is the Lie-Algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$. This is true since $\mathfrak{k} \times \mathfrak{so}(3)$ is the lie algebra of the compact group $\mathrm{SO}(2) \times \mathrm{SO}(3) < \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(3) < \mathrm{GL}(\mathfrak{g})$. Note that we were forced to take the Lie-algebra of a compact group as the second factor ($\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ would not have worked, but $\mathfrak{so}(3) \times \mathfrak{so}(3)$ would have).

Now one can calculate the Killing form

$$A = \begin{pmatrix} -8 & 0 & 0 & & & \\ 0 & 8 & 0 & & 0 & \\ 0 & 0 & 8 & & & \\ & & & -2 & 0 & 0 \\ & 0 & & 0 & -2 & 0 \\ & & & 0 & 0 & -2 \end{pmatrix}$$

in the basis

$$e_1 = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right), e_2 = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), e_3 = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

$$e_4 = \left( 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right), e_5 = \left( 0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right), e_6 = \left( 0, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

of $\mathfrak{g} = \mathfrak{k} \times 0 \oplus \mathfrak{p} \times 0 \oplus 0 \times \mathfrak{so}(3) = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$. The first factor (= Killing form of $\mathfrak{sl}(2, \mathbb{R})$) can be quite quickly computed by hand. As for the second factor (= Killing form of $\mathfrak{so}(3)$) one can notice that $\mathfrak{so}(3)$ is the Lie algebra of a compact semisimple group and hence its Killing form is negative definite.

Since $\mathfrak{i} = \mathfrak{p} \times 0 =< e_2, e_3 > \times 0$, we see that (cfr. definition of $\mathfrak{i}_0, \mathfrak{i}_+, \mathfrak{i}_-$ in the proof of Theorem II.27)

$$\mathfrak{i}_0 = 0 \times 0, \quad \mathfrak{i}_- = \mathfrak{p} \times 0, \quad \mathfrak{i}_+ = 0 \times 0.$$

Therefore

$$\mathfrak{u}_- = [\mathfrak{i}_-, \mathfrak{i}_-] = \mathfrak{k} \times 0, \quad u_+ = [\mathfrak{i}_+, \mathfrak{i}_+] = 0.$$

Now $\mathfrak{u}_+ := [\mathfrak{i}_+, \mathfrak{i}_+] = 0$ and $\mathfrak{u}_- := [\mathfrak{i}_-, \mathfrak{i}_-] = \mathfrak{k} \times 0$. The remaining orthogonal complement in $\mathfrak{u}$ is $\mathfrak{u}_0 = 0 \times \mathfrak{so}(3) \neq 0$. So we have found a OSLA with $\mathfrak{i}_0 = 0$ and $\mathfrak{u}_0 \neq 0$.

The Lie algebra $\mathfrak{g} = \mathfrak{u}_0 \oplus \mathfrak{u}_- \oplus \mathfrak{i}_-$ is of non-compact type.

(b) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_\mathfrak{n} = B_\mathfrak{g}|_{\mathfrak{n}\times\mathfrak{n}}$.

**Solution.** Let us write a basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ of $\mathfrak{g}$, where $e_1, \ldots e_n$ is a basis of $\mathfrak{n}$. Since $\mathfrak{n}$ is an ideal, for $X \in \mathfrak{n}, Z \in \mathfrak{g}$, we have $[X, Z] \in \mathfrak{n}$. Therefore $\mathrm{ad}_\mathfrak{g}(X)$ is of the form

$$\mathrm{ad}_\mathfrak{g}(X) = \begin{pmatrix} \mathrm{ad}_\mathfrak{n}(X) & * \\ 0 & 0 \end{pmatrix}$$

and so for $X, Y \in \mathfrak{n}$ we have

$$\begin{aligned} B_\mathfrak{g}(X,Y) &= \mathrm{tr}(\mathrm{ad}_\mathfrak{g}(X) \circ \mathrm{ad}_\mathfrak{g}(Y)) \\ &= \mathrm{tr} \begin{pmatrix} \mathrm{ad}_\mathfrak{n}(X) \circ \mathrm{ad}_\mathfrak{n}(Y) & * \\ 0 & 0 \end{pmatrix} \\ &= \mathrm{tr}(\mathrm{ad}_\mathfrak{n}(X) \circ \mathrm{ad}_\mathfrak{n}(Y)) \\ &= B_\mathfrak{n}(X,Y). \end{aligned}$$

(c) Find an example of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$, such that $B_\mathfrak{n} \neq B_\mathfrak{g}|_{\mathfrak{n}\times\mathfrak{n}}$.

**Solution.** We consider the Cartan-decomposition $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ and set $\mathfrak{n} := \mathfrak{k}$. We know that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. Taking the basis

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we get

$$\mathrm{ad}_\mathfrak{g}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

(for a derivation see the third exercise class). Let $X = \lambda_1 \cdot e_1, Y = \lambda_2 \cdot e_1 \in \mathfrak{n} = \langle e_1 \rangle$. Then

$$B_\mathfrak{g}(X,Y) = \mathrm{tr}(\mathrm{ad}_\mathfrak{g}(X) \circ \mathrm{ad}_\mathfrak{g}(X)) = \lambda_1 \cdot \lambda_2 \cdot \mathrm{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} = -8 \cdot \lambda_1 \cdot \lambda_2,$$

but

$$B_\mathfrak{n}(X,Y) = \mathrm{tr}(\mathrm{ad}_\mathfrak{n}(X) \circ \mathrm{ad}_\mathfrak{n}(X)) = \mathrm{tr}(0 \cdot 0) = 0$$

since $\mathrm{ad}_\mathfrak{n}(X) = 0$ because $[e_1, e_1] = 0$.

(d) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Show that $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

This implies that $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$.

*Hint*: For connected $G$ and $K < G$, there is an isomorphism

$$K/(K \cap Z(G)) \cong \mathrm{Ad}_G(K)$$

(compare Ex Sheet 2, exercise 5(b)). Use $\mathrm{Lie}(\mathrm{Ad}_G(K)) = \mathrm{ad}_{\mathrm{Lie}(G)}(\mathrm{Lie}(K))$.

**Solution.** By Lie's third theorem, there exist connected, (and simply connected) Lie groups $G_1$ and $G_2$ with $\mathrm{Lie}(G_1) = \mathfrak{g}_1$ and $\mathrm{Lie}(G_2) = \mathfrak{g}_2$. The Lie group $G := G_1 \times G_2$ satisfies $\mathrm{Lie}(G) = \mathfrak{g}_1 \times \mathfrak{g}_2$. Since $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are Lie-subalgebras, there exist $K_1$ and $K_2$ Lie-subgroups of $G_1$ and $G_2$ with $\mathrm{Lie}(K_1) = \mathfrak{k}_1$ and $\mathrm{Lie}(K_2) = \mathfrak{k}_2$. We also have $K := K_1 \times K_2$ with $\mathrm{Lie}(K) = \mathfrak{k}_1 \times \mathfrak{k}_2$.

Now we have the center $Z(G) = Z(G_1) \times Z(G_2)$ and

$$Z(G) \cap K = (Z(G_1) \times Z(G_2)) \cap (K_1 \times K_2) = (Z(G_1) \cap K_1) \times (Z(G_2) \cap K_2),$$

so

$$\begin{aligned} \mathrm{Ad}_G(K) &= K/(Z(G) \cap K) \\ &= (K_1 \times K_2)/(Z(G_1) \cap K_1 \times Z(G_2) \cap K_2) \\ &= K_1/(Z(G_1) \cap K_1) \times K_2/(Z(G_2) \cap K_2) \\ &= \mathrm{Ad}_{G_1}(K_1) \times \mathrm{Ad}_{G_2}(K_2). \end{aligned}$$

Now $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}_1 + \mathfrak{k}_2), \mathrm{ad}_{\mathfrak{g}_1}(\mathfrak{k}_1)$ and $\mathrm{ad}_{\mathfrak{g}_2}(\mathfrak{k}_2)$ are the Lie-algebras of the groups $\mathrm{Ad}_G(K)$, $\mathrm{Ad}_{G_1}(K_1)$ and $\mathrm{Ad}_{G_2}(K_2)$.

So $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ by definition if and only if $\mathrm{Ad}(K)$ is compact which is equivalent to saying $\mathrm{Ad}_{G_1}(K_1)$ and $\mathrm{Ad}_{G_2}(K_2)$ are compact, i.e. both $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are compactly embedded in $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$.

**Exercise 2** (Theorem II.33 - Decomposition of simply connected RSS). (a) Let $H, N \lhd G$ be two normal subgroups. Show that $[N, H] \subset N \cap H$.

**Solution.** Let $nhn^{-1}h^{-1} \in [N, H]$, then $(nhn^{1})h^{-1} \in Hh^{-1} \subset H$ and $n(hn^{-1}h^{-1}) \in nN \subset N$. So $nhn^{-1}h^{-1} \in H \cap N$.

(b) Let $H, N < G$ be connected subgroups. Show that $[N, H]$ is a connected subgroup of $G$.

**Solution.** The map $[\cdot, \cdot] \colon N \times H \to G$ is continuous, since it is a composition of multiplications. The image of connected sets under a continuous map is connected.

(c) Let $M$ be a simply connected Riemannian symmetric space. Then $\mathfrak{g} = \mathrm{Lie}(\mathrm{Iso}(M)^\circ) = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We get corresponding Lie-subgroups $G_0, G_+, G_-$ and their universal covers $\tilde{G}_0, \tilde{G}_+, \tilde{G}_-$. Let $K_0, K_+, K_-$ be the Lie-subgroups associated to $\mathfrak{k}_0, \mathfrak{k}_+, \mathfrak{k}_-$, which come from the Cartan-decomposition of $\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-$.

Show that $(\tilde{G}_0, K_0), (\tilde{G}_+, K_+)$ and $(\tilde{G}_-, K_-)$ are Riemannian symmetric pairs.

**Solution.** Let $\mu \in \{0, +, -\}$. The $\tilde{G}_\mu$ can be assumed to be connected. In the proof of theorem II.33, we have that $\tilde{\psi}|_{K_0 \times K_- \times K_+} : K_0 \times K_- \times K_+ \to p^{-1}(K)$ is a homeomorphism. The product of sets is closed if and only if all the factors are closed, so $K_\mu$ are closed subgroups of $\tilde{G}$ and therefore also of $\tilde{G}_\mu$. Since $\mathfrak{k}_\mu$ are compactly embedded, we get that $\mathrm{Ad}_{\tilde{G}_\mu}(K_\mu)$ are compact.

By the Lie-group-correspondence, since $\tilde{G}$ is simply connected we get $\sigma \colon \tilde{G} \to \tilde{G}$ a unique Lie-group automorphism, such that $\mathrm{D}_e\, \sigma = \theta$. Now (using the pullback of the isomorphism $\psi$), we can restrict $\sigma$ to $\sigma_\mu \colon \tilde{G}_\mu \to \tilde{G}_\mu$. Since $\theta_\mu \colon \mathfrak{g}_\mu \to \mathfrak{g}_\mu$ is an involution, so is $\sigma_\mu$ (they are not the identity).

It remains to show that $(\tilde{G}_\mu^{\sigma_\mu})^\circ \subset K_\mu \subset \tilde{G}_\mu^{\sigma_\mu}$. Let $X \in \mathfrak{k}_\mu$. Then $\exp(X) \in \tilde{G}_\mu$. We have that $\sigma_\mu(\exp(X)) = \exp(\theta_\mu X) = \exp(X)$. So for all $g \in K_\mu$ in a small neighborhood of $e$, we have $\sigma_\mu(g) = g$. Since a neighborhood generates the connected group $K_\mu$, we can write elements $g \in K_\mu$ as a product $g = g_1 \cdot \ldots \cdot g_n$ and we get $\sigma_\mu(g) = \sigma_\mu(g_1) \cdot \ldots \cdot \sigma_\mu(g_n) = g$. So $K_\mu \subset \tilde{G}_\mu^{\sigma_\mu}$.

Now we consider a neighborhood $V \subset \exp(\mathfrak{g}_\mu)$ of $e$ of $\tilde{G}\mu$. Let $\exp(tX) \in V \cap (\tilde{G}_\mu^{\sigma_\mu})^\circ$ for $t \in (-\varepsilon, \varepsilon)$. Then $\exp(tX) = \sigma_\mu(\exp(tX)) = \exp(t\theta_\mu(X))$, so (taking the derivative) we get $X = \theta_\mu(X)$, i.e. $X \in \mathfrak{k}_\mu$ and thus $V \cap (\tilde{G}_\mu^{\sigma_\mu})^\circ \subset K_\mu$. Now since $(\tilde{G}_\mu^{\sigma_\mu})^\circ$ is connected, the elements are generated by elements in $K_\mu$, i.e. $(\tilde{G}_\mu^{\sigma_\mu})^\circ \subset K_\mu$.

We conclude that $(\tilde{G}_\mu, K_\mu)$ are Riemannian symmetric pairs for $\{0, -, +\}$.

**Exercise 3** (Complexification and Killing form)**.** Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let $K_0, K$ and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

(a) $K_0(X, Y) = K(X, Y)$ for all $X, Y \in \mathfrak{l}_0$;

(b) $K^{\mathbb{R}}(X, Y) = 2 \cdot \Re(K(X, Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

**Solution.** The first relation is obvious. For the second let $\mathcal{B} := \{X_i : i = 1, \ldots, n\}$ be a basis of $\mathfrak{l}$. Let $X, Y \in \mathfrak{l}$. Then we may write

$$\mathrm{ad}(X)\,\mathrm{ad}(Y)(X_i) = \sum_{j=1}^{n} \alpha_{ij} \cdot X_j, \qquad i = 1, \ldots, n, \tag{1}$$

for some complex numbers $\alpha_{ij} = \beta_{ij} + i \cdot \gamma_{ij} \in \mathbb{C}$. Denote by $A, B, C$ the $n \times n$-matrices with entries $\alpha_{ij}$, $\beta_{ij}$, $\gamma_{ij}$, respectively. Then $A$ is the matrix representation of $\mathrm{ad}(X)\,\mathrm{ad}(Y)$ with respect to the basis $\mathcal{B}$

$$M_{\mathcal{B}}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) = A = B + iC$$

and $B, C$ are the real, imaginary parts of $A$. Now, consider the basis $\mathcal{C} = \{X_1, \ldots, X_n, iX_1, \ldots, iX_n\}$ of $\mathfrak{l}^{\mathbb{R}}$. Then

$$\mathrm{ad}(X)\,\mathrm{ad}(Y)(iX_i) = \sum_{j=1}^{n} -\gamma_{ij} \cdot X_j + \sum_{j=1}^{n} \beta_{ij} \cdot (iX_j), \qquad i = 1, \ldots, n, \tag{2}$$

and with (**??**) we obtain that the matrix representation of $\mathrm{ad}(X)\,\mathrm{ad}(Y)$ with respect to the basis $\mathcal{C}$ is given by

$$A' := M_{\mathcal{C}}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Thus

$$2\Re K(X,Y) = 2\Re(\mathrm{tr} A) = 2B = \mathrm{tr} A' = K^{\mathbb{R}}(X,Y).$$