Solution Exercise Sheet 5

Exercise 1. Exhibit an explicit isomorphism between the two real Lie algebras $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$.

Hint: Consider the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2,\mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

Solution. Every element $v \in V = \{v \in \mathbb{C}^{2\times 2} : \bar{v}^t = -v\}$ can be written as

$$v = \begin{pmatrix} i(x_1 - x_3) & -x_2 + ix_4 \\ x_2 + ix_4 & i(x_1 + x_3) \end{pmatrix}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. We compute

$$q(v) = \det(v) = -x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

It is readily verified that the given action of $\mathrm{SL}(2,\mathbb{C})$ on V is well-defined and preserves q. Indeed,

$$q(g.v) = \det(gv\bar{g}^t) = \det(g)\det(v)\det(\bar{g})^t = \det(v)$$

for every $g \in \mathrm{SL}(2,\mathbb{C})$ and every $v \in V$.

Thus we obtain a Lie group homomorphism $\varphi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^\circ$, $\varphi(g)(v) = g.v$. It is easy to check that $\{\pm I\} \subseteq \ker\varphi$. Further, if $\varphi(g) = I$ then in particular

$$g\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\bar{g}^t = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$
$$g\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\bar{g}^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$g\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\bar{g}^t = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and it is elementary to deduce that $g = \pm I$. Hence, $\ker\varphi = \{\pm I\}$ and in particular φ is injective on a neighbourhood of I. Because it is a Lie group homomorphism and therefore has constant rank, its differential $d\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ is injective. Both Lie algebras have real dimension 6 such that $d\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ gives indeed the sought for isomorphism.

Exercise 2 (Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$)**.** Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

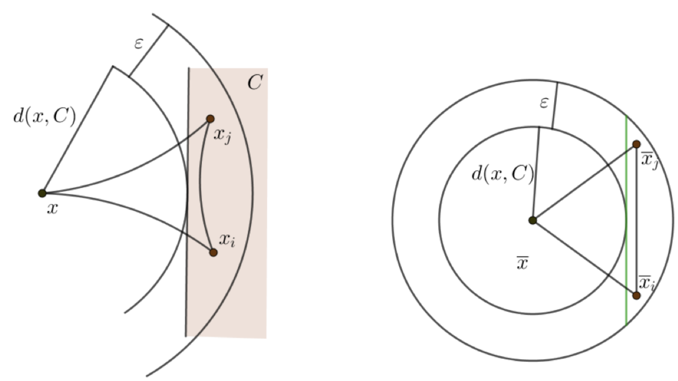
Solution. Recall that we have seen in the lecture that $(\mathrm{SO}(n{+}1), \mathrm{SO}(n), \sigma)$ and $(\mathrm{SO}(1,n)^\circ, \mathrm{SO}(n), \sigma)$ are Riemannian symmetric pairs where $\sigma(g) := I_{1,n} g I_{1,n}$ in both cases. Further we have seen that the associated symmetric spaces $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are isometric to the n-sphere $\mathbb{S}^n$ and (real) hyperbolic n-space $\mathbb{H}^n$. (These are Example(3) after Corollary II.17 and exercise 1 of Exercise Sheet 3, respectively).

These have $(\mathfrak{so}(n+1),\zeta)$ and $(\mathfrak{so}(1,n),\zeta)$ as orthogonal symmetric Lie algebras, respectively, where $\zeta(X) = d\sigma(X) = I_{1,n}XI_{1,n}$ in both cases.

We have also seen in the lecture that the orthogonal symmetric Lie algebras $(\mathfrak{so}(p+q),\zeta_{p,q})$ and $(\mathfrak{so}(p,q),\zeta_{p,q})$ are dual to each other for all $p,q \geq 1$ where $\zeta_{p,q}(X) = I_{p,q}XI_{p,q}$ in both cases. Thus for $p=1, q=n$ we obtain the assertion.

Exercise 3 (CAT(0) spaces)**.** Let (X,d) be a complete CAT(0) space and $\emptyset \neq C \subseteq X$ be a convex closed subset of X. Prove that for every $x \in X$ there exists a unique point $p_C(x) \in C$ such that $d(x,p_C(x)) \leq d(x,y)$ for any $y \in C$.

Solution. Let x be the point that we want to project on C. We consider a sequence of points x_i with $d(x,x_i) \to d(x,C)$ as $i \to \infty$. We want to show that x_i is a Cauchy-sequence. So let $\varepsilon > 0$. There exists an $N > 0$ such that $d(x,x_i) \leq d(x,C)+\varepsilon$ for all $i \geq N$. Consider two points x_i, x_j with $i,j \geq N$. Now consider the comparison triangle $\overline{\Delta}(\overline{x}\overline{x}_i\overline{x}_j)$ of the triangle $\Delta(xx_ix_j)$. This is visualized in figure . Since C is convex, all points on the geodesic between x_i and x_j lie in C, so in the comparison triangle they also need to lie in the annulus between $d(x,C)$ and $d(x,C)+\varepsilon$. A calculation in $\mathbb{R}^2$ shows that such a straight line segment (green line in the figure) can have at most size $2\sqrt{d(x,C)+\varepsilon^2 - d(x,C)^2}$, therefore also $d(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ can have at most this distance and as ε goes to 0, so does the distance $d(x_i,x_j)$.



We have shown that $\{x_i\}$ is a Cauchy sequence, so since the space is complete, there exists a limit point, which we call $\pi(x)$. Since C is closed and all $x_i \in C$, also $\pi(x)$ is in C. By construction $d(x,\pi(x)) = d(x,C)$. We have to show uniqueness:

Let y and y' be two points with minimal distance $d(x,y) = d(x,y') = d(x,C)$. Consider the comparison triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{y}')$. Since $d(\overline{x},\overline{y}) = d(x,y) = d(x,y') = d(\overline{x},\overline{y}')$, $\overline{\Delta}$ is isosceles. Now the midpoint z of y and y' on the unique geodesic between y and y' is in C, since C convex. We also have $\overline{z}$ on the line-segment from $\overline{y}$ to $\overline{y}'$. If $y \neq y'$, then $\overline{z} \neq \overline{y}$ is closer to $\overline{x}$ than $\overline{y}$, i.e. $d(\overline{z}, \overline{x}) < d(\overline{y}, \overline{x})$, thus by the CAT(0)-property also $d(x, z) \leq d(\overline{x}, \overline{z}) < d(\overline{y}, \overline{x}) = d(x, y)$, but that is impossible since $z \in C$ and $d(x, z)$ is the minimal distance from x to all points in C. We conclude that $y = y'$ and thus the projection π_C is well-defined.