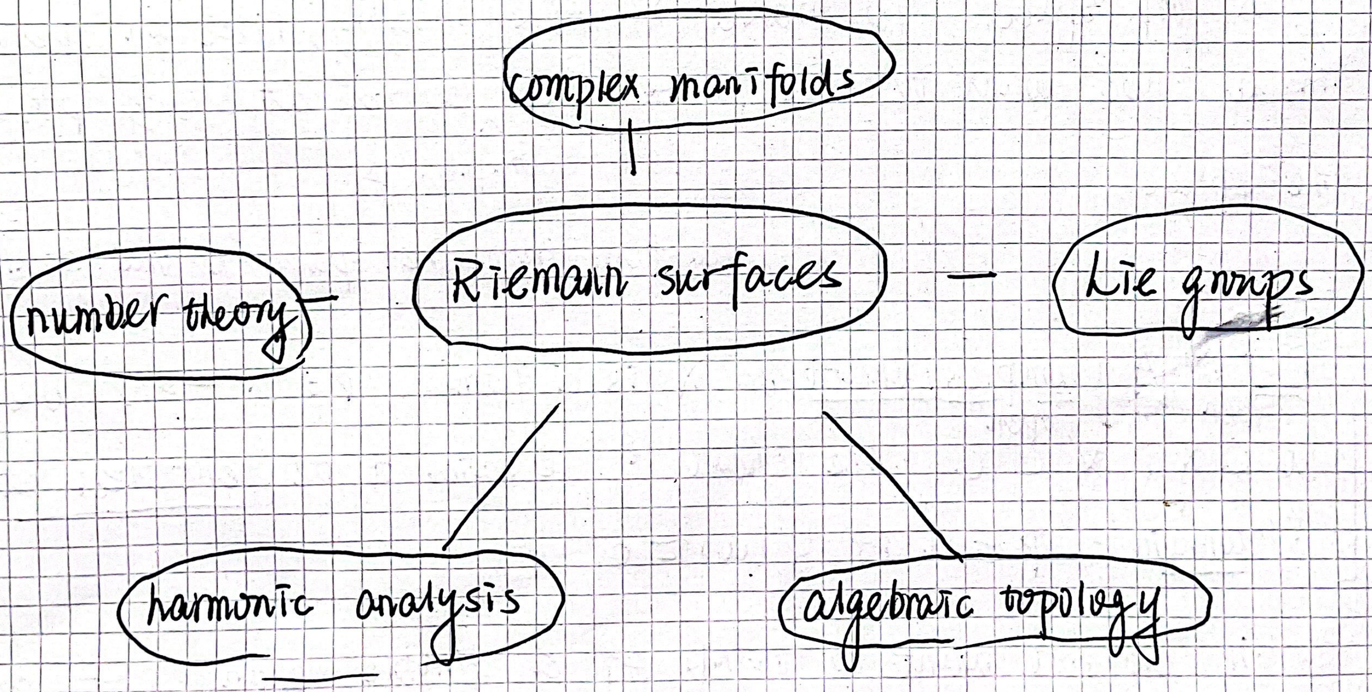
Riemann Surfaces

0. Intro.

Roughly speaking, Riemann surfaces are spaces that locally look like some complex plane $\mathbb{C}$, and on which one can do complex analysis.

The theory of Riemann surfaces lies in the intersection of many important areas of mathematics. Aside from being an important field of study in ~~its~~ its own right, it has long been a source of inspiration, intuition, and examples for many branches of mathematics, e.g., complex manifolds, Lie groups, (algebraic) number theory, harmonic analysis, algebraic topology, ...



Three main viewpoints of compact Riemann surfaces

(1) Complex manifold, as 1-dim. conn. complex mfds
(2) algebro-geometric, as alg. curves
(3) number-theoretic, as function fields (finite ext. of $\mathbb{C}(t)$)

- analysis, differential geometry, topology
- (curvature, characteristic classes, elliptic operators)

  commutative algebra.

  (ideal theory in polynomial rings, valuation theory)

- algebraic number theory.

These are closely related & to some extent ~~equat~~ equivalent.

Btw, the most important result for compact Riemann surfaces is RR, which would be the mainstream of the first half of our seminar.

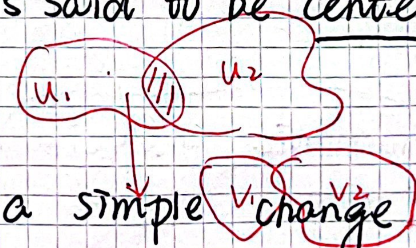
1. Definition & first examples

X top. space. To make X look locally like an open in $\mathbb{C}$, we want to have a ~~c~~ local complex coord. at every pt. of the space.

a fcn. from the space to the std. --

Def: A (complex) chart on X is a homeo. $\varphi : U \to V$, where $\underset{\text{open}}{U} \subset X$, $\underset{\text{open}}{V} \subset \mathbb{C}$. The chart φ is said to be centered at $p \in U$ if $\varphi(u) = 0$.

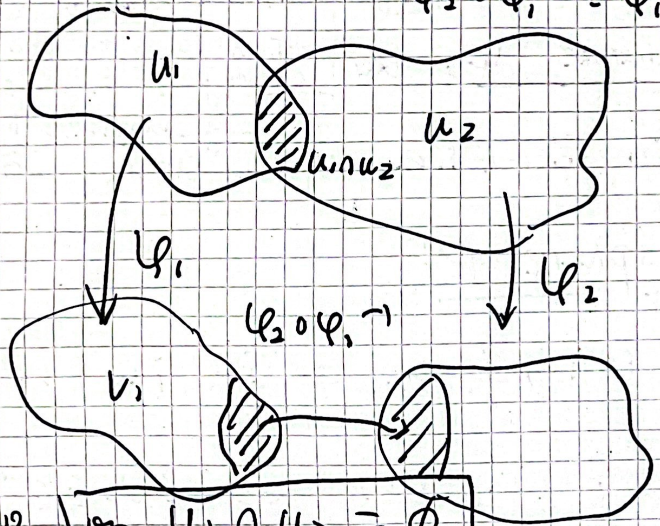
domain



We do not want to think of a simple change of coord. as imposing an essentially different str. on the open set in question. E.g. two charts should not produce different answers when we ~~a~~ get around to ask questions about loc. fcn & forms on the domain.

Def: ~~Tt~~ Two charts $\varphi_1 : U_1 \to V_1$ & $\varphi_2 : U_2 \to V_2$ on X are (holomorphically) compatible if either $U_1 \cap U_2 = \emptyset$ or the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$



bi-
is holo. $\boxed{\text{or } U_1 \cap U_2 = \emptyset}$!!

For X to look loc. like $\mathbb{C}$ everywhere, we must have complex charts around every pt of X ~~&~~ & we want these charts to be compatible.

Def: A (complex) atlas ~~∋~~ $\mathcal{A}$ on X is a collection

$$\mathcal{A} = \{\varphi_i : U_i \to ~~\mathbb{C}~~ V_i\}_{i \in I}$$

compatible of charts ~~and which~~ that cover X, i.e., $X = \bigcup_{i \in I} U_i$.

It may well be the case that two different atlases give the same local notions of complex analysis on a Riemann surface. I.p., this will happen when every chart of one atlas is ~~cp~~ compatible with every chart of the other atlas.

Def: Two complex atlas $\mathcal{A}$ & $\mathcal{A}'$ are (analytically) equival. if $\iff$ $\mathcal{A} \cup \mathcal{A}'$ is also an atlas.

Rmk / Def: Zorn's lemma argument $\rightsquigarrow$ every complex atlas is cont. in a unique max'l one. — complex str.

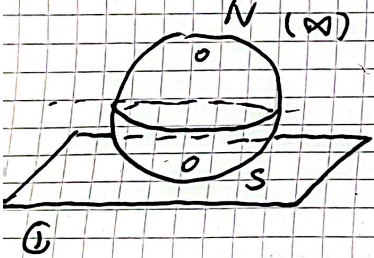
Def: A Riemann surface is a (second countable), Hausdorff conn. top. space X together with a complex str. (1)

Rmk: A R.s. is a sm. real 2-mfd. ~~&~~ that is oriented. If X is cpt., the topologically X is a sphere w/ g handles by ···

E.g. (a) The complex plane $\mathbb{C}$, ~~do~~

(b) ~~Da~~ Domains in $\mathbb{C}$ (i.e., open, conn. subsets)

(c) The Riemann sphere $\mathbb{P}^1$



$$\mathbb{P}^1 := (\mathbb{C}^2 \setminus \{0\})/\sim \; = \mathbb{C} \cup \{\infty\}$$

$$(z_1, z_2) \sim (w_1, w_2) \iff \exists \lambda \in \mathbb{C}^\times \text{ s.t. } w_i = \lambda z_i, \; i = 1, 2.$$

$$p = [z_1 : z_2]$$

① ~~P~~ $z_2 \neq 0 \rightsquigarrow p = [z_1/z_2 : 1] \rightsquigarrow \mathbb{C}$

② $z_2 = 0 \rightsquigarrow p = [1 : 0] \rightsquigarrow \infty$.

"~~One pt. compactification of $\mathbb{C}$~~" Opens are the usual opens $U \subset \mathbb{C}$ & sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a cpt set $K \subset \mathbb{C}$.

$$\varphi_0 : U_0 = \{[z : w] : z \neq 0\} \cong \mathbb{C} \text{ open}, \quad \longmapsto w/z$$

$$\varphi_1 : U_1 = \{[z : w] : w \neq 0\} \cong \mathbb{C} \text{ open}, \quad \longmapsto z/w$$

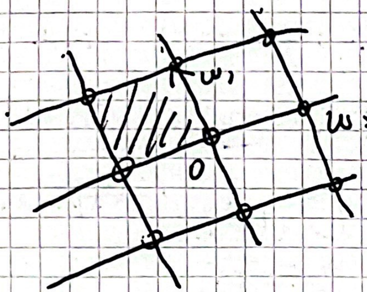
$$\varphi_1 \circ \varphi_0^{-1} : \mathbb{C}^\times \to \mathbb{C}^\times, \quad z \longmapsto z^{-1}$$

Rmk: Another way is to use the spherical projection.

$$x^2 + y^2 + w^2 = 1 \qquad \varphi_1 : (x, y, w) \longmapsto \frac{x}{1-w} + i\frac{y}{1-w}$$

$$\varphi_2 : (x, y, w) \longmapsto \frac{x}{1+w} - i\frac{y}{1+w}$$

d) complex tori. Given a lattice $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ in $\mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly indep. over $\mathbb{R}$. Consider the quotient space



$$\mathbb{C}/\Gamma$$

(two pts z & z' are equiv. if ~~&~~ $z - z' \in \Gamma$.

and the canonical proj.

$$\pi : \mathbb{C} \to \mathbb{C}/\Gamma.$$

Claim: There exists a complex str. on $\mathbb{C}/\Gamma$ s.t. π is holo.

① Topology: The quotient top. i.e., π is open.

Hausdorff ✓ cpt ✓ (covered by the cpt. parallelogram

$$P := \{ \lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in [0,1] \}$$

② Complex str. Let $V \overset{\text{open}}{\subset} \mathbb{C}$ be s.t. no two pts in V are equiv. under Γ. Then $U := \pi(V)$ is open & $\pi|_V : V \to U$ is a homeo. $\overset{\text{inverse}}{\rightsquigarrow}$ $\varphi : U \to V$. Let $\mathcal{A}$ be the set of all charts obtained in this way.

WTS: Any two charts $\varphi_i : U_i \to V_i$ are compatible

$$\psi = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

$$z$$

$$\pi(\psi(z)) = \varphi_1^{-1}(z) = \pi(z)$$

$\Rightarrow \psi(z) - z \in \Gamma$

Γ is discrete & ψ cts.

$\Rightarrow \psi(z) - z$ is const. $\boxed{\text{locally const.}}$

$\Rightarrow \psi$ holo. ✓

2 Fcns & maps

X Riemann surface. $p \in X$, f complex-valued fcn. defined in a nbhd. W of p.

Def: f is holo. at p ~~/resp. mero.~~ /resp. mero. if $\forall$ chart $\varphi: U \to V$ w/ $p \in U$, s.t. $f \circ \varphi^{-1}$ is holo. at $\varphi(p)$. f is holo. in W if it is holo. at every pt of W. --- $\mathcal{O}(W)$. ~~/resp. mero~~ ~~$\mathcal{M}(W)$~~

To understand Riemann surfaces, we need to study three things in particular

- Maps from X to other (simpler) Riemann surfaces
- Maps from --- to X
- * Holo. vector bundles on X.

Def: X, Y Rieman surfaces. A cts mapping $f: X \to Y$ is holo. at $p \in X$ if $\forall$ pair of charts $\varphi_1: U_1 \to V_1$ on X & $\varphi_2: U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the mapping

$$\varphi_2 \circ f \circ \varphi_1^{-1}: V_1 \to V_2$$

is holo. A mapping $f: X \to Y$ is biholo. if it is bij. and both f & f^{-1} are holo. Two Riemann surfaces X and Y are call iso. if $\exists$ a biholo. ---.

Rmk: a) Holo. fcns /mero. / maps are well behaved under sum, product, composition. I.p., $\mathcal{O}(W)$ is a $\mathbb{C}$-alg. ~~$\mathcal{M}(W)$ is a field if W is conn.~~

b) The condition in definitions above does not have to be verified for all charts (resp. pairs of charts), it suffices to check for your favoriate one.

c) Given $f: X \to Y$ holo., it induces a map

$f^*: \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$, $f^*(\varphi) = \varphi \circ f$

for any $V \subset Y$ open. If $g: Y \to Z$ is another holo. mapping, then $(g \circ f)^* = f^* \circ g^*$. "Pull-back"

~~Reminder~~: Classification of singularity: f holo. in a punctured nbhd. of $p \in X$. & R. Removable sing. - Thm

- removable singularity $\Leftrightarrow$ f is loc. bdd.
- pole $\Leftrightarrow$ $\lim_{\substack{z \to z_0 \\ z \neq z_0}} |f(z)| = +\infty$ $\boxed{\text{Laurent series}}$
- essential singularity $\Leftrightarrow$ f oscillates at p.

Thms inherited from complex analysis.

① baby version

Thm. (Discreteness of zeros & poles) $f: X \to \mathbb{C}$ mero. fcn not id. 0.
Then the zeros and poles of f form a discrete subset of X. I.p. X cpt $\Rightarrow$ f has finitely many zeros & poles. (---- preimage)

② Thm (Identity Thm) f, g mero. maps $X \to Y$, suppose $f = g$ on a subset $S \subset W$ which has a limit pt in W, $W \subset$ conn then $f = g$ on W.

③ Thm (Open mapping thm) Any holo. map between ~~on a~~ ~~compact~~ R. S is open.

④ Thm (Maximum modulus principle.) $f: W \subset X$ open, conn. Suppose $\exists p \in W$ s.t. $|f(x)| \geq |f(p)|$, $\forall x \in W$. then f is const. on W.

Cor: $f: X \to Y$ non-const., X cpt $\Rightarrow$ Y cpt. f surj.

Cor: $f: X \to Y$ holo. inj. $\Rightarrow$ f biholo. onto $f(X)$.

Mero. fcns ~~on~~ & holo. maps to $\mathbb{P}^1$

$f \in \mathcal{M}(X)$. At a pole of f, the nat'l value is ∞, so we may define

$$\tilde{f} : X \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \text{ by}$$

$$x \mapsto \begin{cases} f(x) \in \mathbb{C}, & \text{if } x \text{ not pole} \\ \infty, & \text{if } x \text{ is a pole.} \end{cases}$$

Thm. This construction induces a 1-1 corresp. between

$$\{\text{mero. fcns on } X\} \longleftrightarrow \{\text{holo. maps } \tilde{f} : X \to \mathbb{P}^1\}$$

A local property of holo. maps

Thm (local normal form) $f : X \to Y$ non-const. holo. map def. at $p \in X$. Then $\exists!\, m \geq 1$ s.t. $\forall\, \varphi_2 : U_2 \to V_2$ on Y centered at $f(p)$, $\exists$ chart $\varphi_1 : U_1 \to V_1$ centered at p, s.t.

$$\varphi_2 \circ f \circ \varphi_1^{-1}(z) = z^m$$

Similar to the rank thm in the theory of (real) sm. mfds, a holo. map between R. S. ~~also have a~~ can be placed into a particularly simple canonical form by a loc. change of coords. Essentially, it looks like a power map.

This will be crucial when we discuss global properties of holo. maps, such as the degree.

Scanned with CamScanner

Pf: Fix a chart φ_2 on Y centered at $F(p)$, and choose any chart ~~ψ~~ $\psi: U \to V$ on X centered at p. Then the Taylor series for the fcn. $T(w) = \varphi_2 \circ f \circ \psi^{-1}(w)$ is of the form

$$T(w) = \sum_{i=m}^{\infty} c_i w^i,$$

with $c_m \neq 0$. $m \geq 1$ ($T(0) = 0$) $\Rightarrow T(w) = w^m S(w)$ for some holo. fcn $S(w)$ w/ $S(0) \neq 0$.
(near $w=0$)

$\Rightarrow \exists\ R(w)$ near 0. s.t. $R(w)^m = S(w)$

$\Rightarrow T(w) = (w R(w))^m$. Let $\eta(w) = w R(w)$

$\eta'(0) \neq 0 \Rightarrow \eta$ is inv. near 0. Now, the chart

$$\varphi_1 = \eta \circ \psi$$

on X centered at p is as desired since

$$\begin{aligned} \varphi_2 \circ F \circ \varphi_1(z) &= \varphi_2 \circ f \circ \psi^{-1} \circ \eta^{-1}(z) \\ &= T \circ \eta^{-1}(z) \\ &= \eta^m \circ \eta^{-1}(z) \\ &= z^m. \end{aligned}$$

Clearly m is unique. $\square$

3. Doubly periodic ~~mero.~~ fcns

$$\wp = \wp_P(z) = \frac{1}{z^2} + \sum_{w \in P}{}' \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

$$\wp' = -2 \sum_{w \in P} \frac{1}{(z-w)^3}$$

(a) $\wp, \wp'$ are well-defined.

<u>Lem</u>: $\sum'_{w \in P} |w|^{-\alpha} < \infty \iff \alpha > 2$.

Pf: $\sum'_{w \in P} |w|^{-\alpha}$

$$= \sum_{(m,n) \neq (0,0)} |m w_1 + n w_2|^{-\alpha}$$

$$= \sum_{(m,n) \neq (0,0)} \left(|m w_1 + n w_2|^2 \right)^{-\alpha/2} \quad (*)$$

<u>Claim</u>: $\exists\ \delta_1, \delta_2 > 0$, s.t.

$$\delta_1 \leq \frac{|m w_1 + n w_2|^2}{m^2 + n^2} \leq \delta_2$$

To see this, let

$$F(x,y) = \frac{|x w_1 + y w_2|^2}{x^2 + y^2} \quad (x,y) \neq (0,0)$$

it is homogeneous in x, y of deg 0

$\rightsquigarrow$ it is a cts fcn. on $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ ← compact

$\Rightarrow$ has maximum δ_2 & minimum δ_1.

So

$$(*) \sim \sum_{(m,n) \neq (0,0)} (m^2 + n^2)^{-\alpha/2}$$

(i.e., we reduce to the case $\Lambda = \mathbb{Z}i \oplus \mathbb{Z}$) $\iff \alpha > 2$

Cauchy integral test

$$\sim \int_{x^2+y^2 \geq 1} \frac{dx\,dy}{(x^2+y^2)^{\alpha/2}} = \int_0^{2\pi} \int_1^{\infty} \frac{r\,dr\,d\varphi}{r^{\alpha}} = 2\pi \int_1^{\infty} \frac{dr}{r^{\alpha-1}} < \infty$$

$\Rightarrow \wp, \wp'$ well-defined b.c. when w is large enough

$$\wp \quad \left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right|$$

$$= \left| \frac{2zw - z^2}{w^2 (z-w)^2} \right|$$

$$\sim \frac{1}{|w|^3}$$

$$\wp' \quad \left| \frac{1}{(z-w)^3} \right| \sim \frac{1}{|w|^3}$$

Clearly $\wp, \wp'$ are doubly periodic w.r.t Γ

$\wp$ has poles of order 2 at lattice pts of Γ

$\wp'$ —— 3 - — ——

and no other poles.

Thm (Liouville)

Let f be a ~~holo.~~ mero. doubly periodic fcn w.r.t. Γ, then.

① If f is holo. then f is const.

(b.c. f is a fcn. on the cpt fund. parallelogram P

② $\Rightarrow f$ is bdd. $\Rightarrow f$ is const.

* ② The number of zeros & poles in a fund. para. is equal (counting w/ multiplicity)

So for b)

WTS: Around $z=0$

$$f(z) = \underbrace{-\frac{1}{z^2} + c_2 z^2 + \cdots}_{g(z)\ \text{(doubly periodic)}}$$

Consider

$$g(z) - f(z) =: h(z)$$

it has no poles. and thus holo. $\Rightarrow$ $h(z)$ is const.

$$h(0) = 0 \Rightarrow h(z) \equiv 0 \Rightarrow g(z) = f(z). \checkmark$$