10 Harmonic forms and Functions and differential forms with prescribed principal parts

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Topic: Complex conjugation, the $*$-operator, Harmonic differential forms, scalar prduct on $\mathcal{E}^{(1)}(X)$, prove that $p_a(X) = b_1(X)$ for any Riemann Surface (pages 153-158 of [2]).
Time permitting cover also the section 'Functions and differential forms with prescribed principal parts': Mittag-Leffler distributions of meromorphic functions, the Wronskian determinant, differential forms with prescribed principal parts (pages 146-152 of [2]).

Harmonic Differential Forms (§19)

<u>Def</u> (19.1) Let $\omega \in \mathcal{E}^{(1)}(X)$ be a differentiable 1-form on the Riemann surface X. The complex conjugation of functions induces a **conjugate complex differential 1-form $\overline{\omega}$** $\in \mathcal{E}^{(1)}(X)$: Locally $\omega = \sum f_i dg_i$, where the functions f_i and g_i are differentiable (b.c. ω is differentiable).

Thus $\overline{\omega} = \sum \overline{f_i} d\overline{g_i}$.

A 1-form $\omega \in \mathcal{E}^{(1)}(X)$ is said to be **real** if $\omega = \overline{\omega}$.

The real part of a differential form ω is defined by

$$\mathrm{Re}(\omega) := \frac{1}{2}(\omega + \overline{\omega}).$$

- ω is real $\Leftrightarrow$ $\omega = \mathrm{Re}(\omega)$

- If c is a curve on X (and ω is closed or c piecewise continuously differentiable), then $\overline{\int_c \omega} = \int_c \bar{\omega}$ and thus $\mathrm{Re}\left(\int_c \omega\right) = \int_c \mathrm{Re}(\omega)$.

If $\omega \in \Omega(X)$ is a holomorphic 1-form, then $\bar{\omega}$ is called **anti-holomorphic.**

Denote the space of all anti-holomorphic 1-forms on X by $\overline{\Omega}(X)$.

Def: (1.92) Any 1-form $\omega \in \mathcal{E}^{(1)}(X)$ may be uniquely decomposed as $\omega = \omega_1 + \omega_2$
↳ B.c. def. of differentiable

where $\omega_1 \in \mathcal{E}^{1,0}(X)$, $\omega_2 \in \mathcal{E}^{0,1}(X)$.

Set $\boldsymbol{*\omega} := i(\bar{\omega}_1 - \bar{\omega}_2)$.

The mapping $\boldsymbol{*}: \mathcal{E}^{(1)}(X) \to \mathcal{E}^{(1)}(X)$ is an $\mathbb{R}$-linear isomorphism which maps $\mathcal{E}^{1,0}(X)$ onto $\mathcal{E}^{0,1}(X)$ and vice versa.

→ $\mathbb{R}$-linear: $*(\omega + k\delta) = i\left(\overline{\omega_1 + k\delta_1} - \left(\overline{\omega_2 + k\delta_2}\right)\right) = i(\bar{\omega}_1 + k\bar{\delta}_1 - \bar{\omega}_2 - k\bar{\delta}_2) = i(\bar{\omega}_1 - \bar{\omega}_2) + ki(\bar{\delta}_1 - \bar{\delta}_2) = *\omega + k*\delta$

→ $\mathcal{E}^{1,0}(X) \Leftrightarrow \mathcal{E}^{0,1}(X)$: $\omega = \underbrace{\psi dz}_{=\omega_1 \in \mathcal{E}^{1,0}(X)} + \underbrace{\varphi d\bar{z}}_{=\omega_2 \in \mathcal{E}^{0,1}(X)} \Rightarrow *\omega = i(\overline{\psi dz} - \overline{\varphi d\bar{z}}) = \underbrace{i\bar{\psi} d\bar{z}}_{\in \mathcal{E}^{0,1}(X)} - \underbrace{i\bar{\varphi} dz}_{\in \mathcal{E}^{1,0}(X)}$

→ complex conjugation has no kernel → inj. + finite dim Endom.

__Properties__ For $\omega \in \mathcal{E}^{(1)}(X)$, $\omega_1 \in \mathcal{E}^{1,0}(X)$, $\omega_2 \in \mathcal{E}^{0,1}(X)$ and $f \in \mathcal{E}(X)$, one has:

(a) $**\omega = -\omega$, $\overline{*\omega} = *\overline{\omega}$,

(b) $d*(\omega_1 + \omega_2) = id'\overline{\omega}_1 - id''\overline{\omega}_2$

$\hookrightarrow$ $d*(\omega_1 + \omega_2) = id(\overline{\omega}_1 - \overline{\omega}_2) = id'\overline{\omega}_1 - id''\overline{\omega_2}$

(c) $*d'f = id''\overline{f}$, $*d''f = id'\overline{f}$,

$\hookrightarrow$ $\underbrace{*d'f}_{\in \mathcal{E}^{1,0}(X)} = *\left(\frac{\partial f}{\partial z} dz\right) = i\left(\overline{\frac{\partial f}{\partial z} dz}\right) = i\frac{\partial \overline{f}}{\partial \overline{z}} d\overline{z} = id''\overline{f}$

(d) $d*df = 2id'd''\overline{f}$.

9.14 elt. properties: $ddf = d'd'f = d''d''f = 0$, $d'd''f = -d''d'f$

$\hookrightarrow$ $d*df = d*(d'f + d''f) = d(*d'f + *d''f) = d(id''\overline{f} - id'\overline{f}) = i(d'd''\overline{f} + d''d''\overline{f} - d'd'\overline{f} - d''d'\overline{f}) = 2id'd''\overline{f}$

__Def__ (19.3) A 1-form $\omega \in \mathcal{E}^{(1)}(X)$ on a RS X is called **harmonic** if

$$d\omega = d*\omega = 0.$$

__Thm__ Let $\omega \in \mathcal{E}^{(1)}(X)$. Then the following conditions are equivalent:

(i) ω is harmonic,

(ii) $d'\omega = d''\omega = 0$,

(iii) $\omega = \omega_1 + \omega_2$ where $\omega_1 \in \Omega(X)$ and $\omega_2 \in \overline{\Omega}(X)$

(iv) $\forall a \in X$ there exists an open neighborhood U of a and a harmonic function f on U s.t. $\omega = df$.

Proof: The equivalence of (i), (ii) and (iii) follows from above. B.c.:

(i) $\Rightarrow$ (iv): Since in particular a harmonic differential form is closed, locally $\omega = df$, where f is differentiable function. Since $0 = d{*}\omega = d{*}df = 2i\,d'd''f$, it follows that f is harmonic.

(iv) $\Rightarrow$ (i): If $\omega = df$ and f is harmonic, then $d\omega = ddf = 0$ and $d{*}\omega = d{*}df = 0$.

Notation: We denote the vector space of all harmonic 1-forms on the RS X by $\mathrm{Harm}^1(X)$. Thus $\mathrm{Harm}^1(X) = \Omega(X) \oplus \overline{\Omega}(X)$.

Thus if X is a compact RS of genus g, then $\dim \mathrm{Harm}^1(X) = 2g$.

Thm (19.4) Every real harmonic 1-form $\sigma \in \mathrm{Harm}^1(X)$ is the real part of precisely one holomorphic 1-form $\omega \in \Omega(X)$.

Proof: • Suppose $\sigma = \omega_1 + \overline{\omega_2}$ with $\omega_1, \omega_2 \in \Omega(X)$. B.c. $\sigma = \omega_1 + \overline{\omega_2} = \overline{\sigma} = \overline{\omega_1} + \omega_2$, it follows that $\omega_1 = \omega_2$. Thus $\sigma = \mathrm{Re}(2\omega_1)$.

• To prove uniqueness, suppose $\omega \in \Omega(X)$ and $\mathrm{Re}(\omega) = 0$. Since locally $\omega = df$, where f is a holomorphic function, it follows that f has constant real part. Then f itself is constant and thus $\omega = 0$.

Assume now X is a compact RS.

Def (19.5) For $\omega_1, \omega_2 \in \mathcal{E}^{(1)}(X)$ let $\boldsymbol{<\omega_1, \omega_2>} := \iint_X \omega_1 \wedge * \omega_2$.

Clearly, the mapping $(\omega_1, \omega_2) \mapsto <\omega_1, \omega_2>$ is linear in the first, and semi-linear in the second argument and $<\omega_2, \omega_1> = \overline{<\omega_1, \omega_2>}$.

We now claim that $<,>$ is positive definite:

↳ Proof: Suppose $\omega \in \mathcal{E}^{(1)}(X)$. With respect to a local chart (U, z), where $z = x + iy$, suppose $\omega = f dz + g d\bar{z}$. Then $*\omega = i(\bar{f} d\bar{z} - \bar{g} dz)$ and

$$\omega \wedge * \omega = i(|f|^2 + |g|^2) dz \wedge d\bar{z} = 2(|f|^2 + |g|^2) dx \wedge dy.$$

This shows that $<\omega, \omega> \geq 0$ and $<\omega, \omega> = 0$ only if $\omega = 0$.

Hence with this scalar product $\mathcal{E}^{(1)}(X)$ becomes a unitary vector space. However it is not a Hilbert space, since it is not complete.

<u>Lemma</u> (19.6) X compact RS.

(a) $d'\mathcal{E}(X)$, $d''\mathcal{E}(X)$, $\Omega(X)$ and $\overline{\Omega}(X)$ are pairwise orthogonal vector subspaces of $\mathcal{E}^{(1)}(X)$.

(b) $d\,\mathcal{E}(X)$ and $*d\,\mathcal{E}(X)$ are orthogonal vector subspaces of $\mathcal{E}^{(1)}(X)$ and $d\mathcal{E}(X) \oplus *d\mathcal{E}(X) = d'\mathcal{E}(X) \oplus d''\mathcal{E}(X)$.

<u>Proof:</u> (a) Since $\mathcal{E}^{1,0}(X)$ and $\mathcal{E}^{0,1}(X)$ are trivially orthogonal, it suffices to show that $d'\mathcal{E}(X) \perp \Omega(X)$ and $d''\mathcal{E}(X) \perp \overline{\Omega}(X)$.

Suppose $f \in \mathcal{E}(X)$ and $\omega \in \Omega(X)$. Then $\omega \wedge *d'f = \omega \wedge (i d''\bar{f}) \overset{\text{b.c. } \omega\wedge d'\bar{f}=0}{=} i\omega \wedge d\bar{f} \overset{d\omega=0 \text{ as } \omega\in\Omega(X)}{=} -i\,d(\bar{f}\omega)$.

Thus $\langle \omega, d'f \rangle = -i \iint_X d(\bar{f}\omega) = 0$ by Thm (10.20)

Similarly one can show $\langle \bar{\omega}, d''f \rangle = 0$.

(b) Suppose $f, g \in \mathcal{E}(X)$. Then $df \wedge *(*dg) = -df \wedge dg \overset{ddg=0}{=} -d(f\,dg)$

Thus $\langle df, *dg\rangle = -\iint_X d(f\,dg) = 0$

The equality $d\mathcal{E}(X) \oplus *d\mathcal{E}(X) = d'\mathcal{E}(X) \oplus d''\mathcal{E}(X)$ follows from (19.2.c) $\square$

BC: $*d'f = i d''\bar{f}$, $*d''f = -i d'\bar{f}$

$\Leftrightarrow$ $d''f = -i * d'\bar{f}$, $d'f = i * d''\bar{f}$

$$\left.\begin{aligned} \Rightarrow\ df &= d'f + d''f \\ *df &= *d'f + *d''f = i d''\bar{f} - i d'\bar{f} \end{aligned}\right\} \Rightarrow \text{"}\subseteq\text{"}$$

$$\left.\begin{aligned} \Rightarrow\ d'f &= df - d''f = df + i * d'\bar{f} \\ d''f &= df - d'f = df - i * d''\bar{f} \end{aligned}\right\} \Rightarrow \text{"}\supseteq\text{"}$$

<u>Corollary:</u> (19.7) On a compact RS X every exact differential form $\sigma \in \mathrm{Harm}^1(X)$ vanishes and every harmonic function $f \in \mathcal{E}(X)$ is constant

<u>Proof:</u> $d\mathcal{E}(X) \perp \mathrm{Harm}^1(X)$ (by Lemma 19.6)

Corollary (19.8) X compact RS, $\sigma \in \mathrm{Harm}^1(X)$, $\omega \in \Omega(X)$. If for every closed curve γ on X one has $\int_\gamma \sigma = 0$ resp. $\mathrm{Re}(\int_\gamma \omega) = 0$, then $\sigma = 0$, resp. $\omega = 0$.

Proof: Since σ (resp. $\mathrm{Re}(\omega)$) is exact by Thm (10.15), the result follows from (19.7) and (19.4).

Theorem 19.9 On any compact RS X there is an orthogonal decomposition

$$\mathcal{E}^{0,1}(X) = d''\mathcal{E}(X) \oplus \overline{\Omega}(X).$$

Proof: Let g be the genus of X. Since $H^1(X,\mathcal{O}) \cong \mathcal{E}^{0,1}(X)/d''\mathcal{E}(X)$ by Dolbeault's Thm (15.14), one has $\dim\left(\mathcal{E}^{0,1}(X)/d''\mathcal{E}(X)\right) = g$.

On the other hand, $\dim \overline{\Omega}(X) = g$ by (17.10). The result now follows from Lemma (19.6.a).

Corollary (19.10) X compact RS, $\sigma \in \mathcal{E}^{0,1}(X)$. The equation $d''f = \sigma$ has a solution if and only if $\iint_X \sigma \wedge \omega = 0$ for every $\omega \in \Omega(X)$.

Proof. $\iint_X \sigma \wedge \omega = 0 \;\; \forall \omega \in \Omega(X) \iff \sigma \perp \overline{\Omega}(X)$ (b.c. $\iint_X \sigma \wedge \omega = -\langle \sigma, \overset{=i\overline{\omega}}{*\omega} \rangle$ as $\omega \in \Omega \subseteq \mathcal{E}^{1,0}$)

$\Rightarrow$ Statement follows from Thm 19.9.

Theorem 19.11 On any compact RS X there is an orthogonal decomposition

$$\mathcal{E}^{(1)}(X) = *d\mathcal{E}(X) \oplus d\mathcal{E}(X) \oplus \mathrm{Harm}^1(X).$$

Proof: Taking complex conjugates in Thm (19.9) one gets

$$\mathcal{E}^{1,0}(X) = d'\mathcal{E}(X) \oplus \Omega(X). \text{ Thus } \mathcal{E}^{(1)}(X) = d'\mathcal{E}(X) \oplus d''\mathcal{E}(X) \oplus d''(X) \oplus \Omega(X) \oplus \overline{\Omega}(X).$$

Hence the Result follows from (19.6).

Theorem 19.12 Suppose X is a compact RS. Then
$$\operatorname{Ker}\left(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X)\right) = d\,\mathcal{E}(X) \oplus \operatorname{Harm}^1(X).$$

Proof: Since $\mathcal{Z}(X) := \operatorname{Ker}\left(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X)\right) \supset d\,\mathcal{E}(X) \oplus \operatorname{Harm}^1(X)$, it suffices (by Thm 19.11) to show $\mathcal{Z}(X) \perp *d\,\mathcal{E}(X)$. Suppose $\omega \in \mathcal{Z}(X)$ and $f \in \mathcal{E}(X)$. Then $\omega \wedge *(*df) = -\omega \wedge df = d(f\omega)$ (bc. $d\omega = 0$)

Hence $\langle \omega, *df \rangle = \iint_X d(f\omega) = 0$.

Corollary (19.13) Suppose X is a compact RS. Then a differential form $\sigma \in \mathcal{E}^{(1)}(X)$ is exact iff for every closed 1-form $\omega \in \mathcal{E}^{(1)}(X)$ one has $\iint_X \sigma \wedge \omega = 0$.

Proof: The given condition is equivalent to $\langle \omega, *\sigma \rangle = 0$ for every closed 1-form ω. But by thm 19.11 this means $*\sigma \in *d\,\mathcal{E}(X)$, i.e. $\sigma \in d\,\mathcal{E}(X)$.

Theorem 19.14 (deRahm-Hodge) Suppose X is a compact RS. Then

$$H^1(X,\mathbb{C}) \cong Rh^1(X) \cong Harm^1(X)$$

Proof: Because of (19.12) this follows directly from deRahm's Thm (15.15). $\to H^1(X,\mathbb{C}) \cong Rh^1(X)$. | Recall: $Rh^1(X) := \dfrac{\ker(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X))}{\operatorname{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X))}$

Rem: Since the sheaf $\mathbb{C}$ of locally constant complex-valued functions on X depends only on the topological structure of X, it follows that $b_1(X) := \dim(H^1(X,\mathbb{C}))$ the first Betti number of X, is a topological invariant.

From (19.14), one has $b_1(X) = 2g = 2p_a(X)$

Weierstrass points $(G, \mathbb{C})$

Def (18.4) Let $f_1, \dots, f_g$ be holomorphic functions on a domain $U \subset \mathbb{C}$. The the **Wronskian determinant of $f_1, \dots, f_g$** is the determinant of the matrix of derivatives $f_k^{(m)}$, where $0 \leq m \leq g-1$, $k \leq g$, i.e.,

$$\mathbf{W(f_1, \dots f_g)} := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_g \\ f_1' & f_2' & \cdots & f_g' \\ \vdots & \vdots & & \vdots \\ f_1^{(g-1)} & f_2^{(g-1)} & \cdots & f_g^{(g-1)} \end{pmatrix}$$

Rem: If $f_1, \dots, f_g$ are linearly independent over $\mathbb{C}$, then the Wronskian determinant is not identically zero. (Can be proved by induction)

Def X compact RS of genus $g \geq 1$ and $\omega_1, \dots, \omega_g$ a basis of $\Omega(X)$. For any coordinate neighborhood (U, z) we can define a holomorphic function $\boldsymbol{W_z(\omega_1, \dots, \omega_z)}$ on U as follows: The 1-forms ω_k may be written $\omega_k = f_k dz$ on U (f_k holomorphic).
Set $W_z(\omega_1, \dots, \omega_g) := W(f_1, \dots, f_g)$ (derivatives of the f_k are taken w.r.t. z)

Thm 18.5 Suppose (U, z) and $(\tilde{U}, \tilde{z})$ are two coordinate neighborhoods on X. Then on $U \cap \tilde{U}$ one has $W_{\tilde{z}}(\omega_1, \dots, \omega_g) = \left(\frac{dz}{d\tilde{z}}\right)^N W_z(\omega_1, \dots, \omega_g)$, where
$N = \frac{g(g+1)}{2}$

↳ Proof p. 150

<u>Rem:</u> If $\tilde{\omega}_1, \dots, \tilde{\omega}_g$ is another basis of $\Omega(X)$, then $\exists\, c_{jk} \in \mathbb{C}$ with $\det(c_{jk}) =: c \neq 0$ s.t. $\omega_j = \sum_k c_{jk} \tilde{\omega}_k$. Then

$$W_z(\omega_1, \dots, \omega_g) = c\, W_z(\tilde{\omega}_1, \dots, \tilde{\omega}_g).$$

$\rightsquigarrow$ The following definition is meaningful, i.e. it doesn't depend on the choice of basis of $\Omega(X)$ nor on the local coordinate.

<u>Def 18.6</u> X compact RS of genus $g \geq 1$. A point $p \in X$ is called a **Weierstrass point**, if for a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ and a coordinate neighborhood (U, z) of p, the Wronskian determinant $W_z(\omega_1, \dots, \omega_g)$ has a zero at p.
The order of this zero is called the **weight of the**

Weierstrass point.

By definition a RS of genus 0, i.e., $\mathbb{P}^1$, doesn't have any Weierstrass points.

<u>Theorem 18.7</u> X compact RS of genus g, $p \in X$. Then there exists a non-constant meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order $\leq g$ at p and is holomorphic on $X \setminus \{p\}$ iff p is a Weierstrass point.

Excursion (for proof)

<u>Def (18.1)</u>: X RS, $\mathcal{U} = (U_i)_{i \in I}$ an open covering of X. A cochain $\mu = (f_i) \in C^0(\mathcal{U}, \mathcal{M})$ is called a **Mittag-Leffler distribution** if the difference $f_j - f_i$ are holomorphic on $U_i \cap U_j$, i.e., $\delta\mu \in Z^1(\mathcal{U}, \mathcal{O})$.

Thus the functions f_i and f_j have the same principal parts on their common domain of definition.

A **solution** of μ is a global meromorphic function $f \in \mathcal{M}(X)$ which has the same principal parts as μ, i.e. $f|U_i - f_i \in \mathcal{O}(U_i)$ $\forall i \in I$. Denote by $\boldsymbol{[\delta\mu]} \in H^1(X, \mathcal{O})$ the cohomology class represented by the cocycle $\delta\mu$.

<u>Thm</u> A Mittag-Leffler distribution μ has a solution iff $[\delta\mu] = 0$. (Proof p. 147)

<u>Thm (15.2)</u> Suppose $\mu \in C^0(\mathcal{U}, \mathcal{M})$ is a Mittag-Leffler distribution of meromorphic functions on the compact RS X. Then μ has a solution iff $\operatorname{Res}(\omega\mu) = 0$ $\forall \omega \in \Omega(X)$. (Proof p. 147)

<u>Rem:</u> If $\omega_1, \dots, \omega_g$ is a basis of $\Omega(X)$, then $\operatorname{Res}(\omega\mu) = 0$ $\forall \omega \in \Omega(X)$ iff $\operatorname{Res}(\omega_k \mu) = 0$ for $k = 1, \dots, g$.

$\square$

Proof of Thm 18.7 Let $\omega_1, \dots, \omega_g$ a basis of $\Omega(X)$ and (U, z) a coordinate neighborhood of p with $z(p) = 0$. We can expand the ω_k in series about p $\quad \omega_k = \sum_{\nu=0}^{\infty} a_{k\nu} z^\nu dz, \quad k = 1, \dots, g$. The function f which we are looking for has a principal part at p of the form $h = \sum_{\nu=0}^{g-1} \frac{c_\nu}{z^{1+\nu}}, \quad (c_0, \dots, c_{g-1}) \neq (0, \dots, 0)$ and thus is a solution of the Mittag-Leffler distribution $\mu = (h, 0) \in C^0(\mathcal{U}, \mathcal{M})$, $\mathcal{U} = (U, X \setminus \{p\})$. So (by 18.2) such f exists iff $\operatorname{Res}(\omega_k, \mu) = 0$ $\forall k \in \{0, \dots, g-1\}$. But $\operatorname{Res}(\omega_k, \mu) = \operatorname{Res}_p(\omega_k h) = \sum_{\nu=0}^{g-1} a_{k\nu} c_\nu$.

So the equations $\operatorname{Res}_p(\omega_k h) = 0$ have a non-trivial solution $(c_0, \dots, c_{g-1})$ iff $\det(a_{k\nu}) = 0$. But this is equivalent to $W_z(\omega_1, \dots, \omega_g)(p) = 0$. $\square$

Theorem 18.8 On a compact RS of genus g the number of Weierstrass points, counted according to their weights, is $(g-1)g(g+1)$

Proof: Suppose (U_i, z_i), $i \in I$, is a covering of X by coordinate neighborhoods. On $U_i \cap U_j$ the function $\psi_{ij} := (dz_j / dz_i)$ is holomorphic and has no zeros. Fix a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$. Let $W_i := W_{z_i}(\omega_1, \dots, \omega_g) \in \mathcal{O}(U_i)$. By Thm 18.5 one has $W_i = \psi_{ij}^N W_j$ on $U_i \cap U_j$, where $N = \frac{g(g+1)}{2}$. Set $D(x) := \operatorname{ord}_x(W_i)$ for $x \in U_i$, this defines the divisor D on X corresponding to the Weierstrass points together with their respective weights. Thus $\deg D$ is the total of weights of the Weierstrass points and the proof is complete once we show $\deg D = (g-1)g(g+1)$. Let D_1 be the divisor of ω_1. Then $\deg D_1 = 2g-2$ by Thm

17.12. (The divisor of a non-vanishing meromorphic 1-form ω on a compact RS of genus g satisfies $\deg(\omega) = 2g-2$) If we set $\omega_1 = f_{1i}\, dz_i$ on U_i, then $D_1(x) = \mathrm{ord}_x(f_{1i})$ for every $x \in U_i$. Moreover $f_{1i} = \varphi_{ij} f_{1j}$ on $U_i \cap U_j$. This (together with the fact that $W_i = \varphi_{ij}^N W_j$ on $U_i \cap U_j$) now implies that $W_i f_{1i}^{-N} = W_j f_{1j}^{-N}$ on $U_i \cap U_j$.

Thus there exists a global meromorphic function $f \in \mathcal{M}(X)$ with $f|U_i = W_i f_{1i}^{-N}$. For the divisor of f one has $(f) = D - ND_1$.

Since $\deg(f) = 0$, it follows that

$$\deg D = N \deg D_1 = \frac{g(g+1)}{2}(2g-2) = (g-1)g(g+1). \qquad \square$$

<u>Corollary 18.9</u> Every compact RS X of genus $g \geq 2$ admits a holomorphic covering mapping $f: X \to \mathbb{P}^1$ having at most g sheets. In particular, every

compact RS of genus 2 is hyperelliptic.

<u>Remark:</u> $g(C)=1 \Rightarrow \exists\ f: C \longrightarrow \mathbb{P}^1$ degree 2 map

(reason : take $f = \wp$ to be the Weierstrass fct)

$g(C)=2 \Rightarrow \exists\ F: C \longrightarrow \mathbb{P}^1$ degree 2 map

(reason : use cor)

the proof of this does not apply to $g(C)=1$ bcs you don't have Weierstrass pts on C.