20. Abel's Theorem

20.1 Functions with Prescribed Divisors

X: Riemann surface. D: divisor on X.

Def: $f \in \mathcal{M}(X)$ is a solution of D if $(f) = D$.

Rmk: Recall 10.22: Any nonconst $f \in \mathcal{M}(X)$ on a compact RS X has, counting multiplicities, as many zeros as poles.

$\Rightarrow$ ~~Only if $\deg D$~~ if X is compact,

only if $\deg D = 0$, then it's possible for D to have a solution.

Def: weak solution of D: $X_D := \{x \in X \mid D(x) \geq 0\}$

Define $f \in \mathcal{E}(X_D)$ s.t.

$\forall a \in X$. $\exists$ a coordinate nbhd (U, z) with $z(a) = 0$

$\psi \in \mathcal{E}(U)$ with $\psi(a) \neq 0$

s.t. $f = \psi z^k$ on $U \cap X_D$, where $k = D(a)$.

Use continuity to extend f to all of X.

Rmk: if f is holomorphic on X_D, then f is a meromorphic solution of D.

Rmk: Two weak solutions f and g of D differ by a factor $\varphi \in \mathcal{E}(X)$ which never vanishes.

Rmk: if f_1 (resp. f_2) is a weak solution of D_1 (resp. D_2) then $f := f_1 f_2$ is a weak solution of $D := D_1 + D_2$

Note: at $a \in X$ s.t. $D(a) \geq 0$ But $D_1(a) < 0$ or $D_2(a) < 0$.

$f_1 f_2$ is not defined, but just use continuity to extend f to these points.

Rmk: Similarly f_1/f_2 is a weak solution of $D_1 - D_2$.

20.2 Logarithmic Differentiation

~~f is a we~~ Let f be a weak solution of D.

$\mathrm{Supp}(D) := \{x \in X \mid D(x) \neq 0\}$

Rmk: the logarithmic derivative df/f is a smooth 1-form on $X \setminus \mathrm{Supp}(D) \subseteq X_D$.

②

Rmk if $a \in \operatorname{Supp} D$. $k = D(a)$

Using the local representation.

$$\Rightarrow \frac{df}{f} = k\frac{dz}{z} + \frac{d\psi}{\psi}.$$

Since $\frac{d\psi}{\psi}$ is differentiable in a nbhd of a.

$\Rightarrow \frac{df}{f}$ is also a smooth 1-form ~~on~~ in a nbhd of a except the point a ~~a nbhd of a.~~

~~Rmk for $\forall$ form $\sigma \in \mathcal{E}^{(1)}(X)$ with compact support~~
~~As in (13.11) ("compact support") (restricted to ER then use 4.10 and stokes' theorem)~~
~~$\iint_X \frac{df}{f} \wedge \sigma$ exists.~~

Rmk ~~$\frac{d''f}{f}$ is differentiable on~~

~~Since~~ By using the local representation,

$$\Rightarrow \frac{d''f}{f} = \frac{d''\psi}{\psi}.$$

~~then, $\frac{d''f}{f}$ is differentiable on all of X.~~

$\boxed{\text{20.3 Lemma}}$

$a_1, \cdots, a_n$ are distinct points on RS X.

$k_1, \cdots, k_n \in \mathbb{Z}$

$D \in \operatorname{Div}(X)$ with $D(a_j) = k_j$. $1 \le j \le n$, $D(x) = 0$ otherwise

f: a weak solution of D.

Then $\forall g \in \mathcal{E}(X)$ with compact support,

$$\frac{1}{2\pi i}\iint_X \frac{df}{f} \wedge dg = \sum_{j=1}^{n} k_j g(a_j)$$

~~(note: this exists, since $g \in \mathcal{E}(X)$ with compact support)~~

Proof: ~~Choose disjoint~~ Since $a_1, \ldots, a_n$ disjoint,

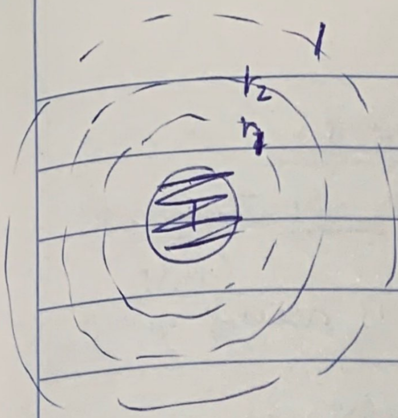
we can choose disjoint coordinate nbhds (U_j, z_j) of a_j with $z(a_j) = 0$ s.t. on U_j. f is written as

$$f = \psi_j z_j^{k} \quad \text{with } \psi_j \in \mathcal{E}(U_j).\ \psi_j(x) \ne 0,\ \forall x \in U_j.$$

w.m.a. $z_j(U_j) \subset \mathbb{C}$ is the unit disk, $\forall j = 1, \ldots, n$

Suppose $0 < r_1 < r_2 < 1$. $\exists \varphi_j \in \mathcal{E}(X)$ s.t.

$\operatorname{supp}(\varphi_j) \subset \{|z_j| < r_2\}$. $\varphi_j|_{\{|z_j| \le r_1\}} = 1$



Let $g_j := \varphi_j g$ for $\forall j = 1, \ldots n$.

$g_0 := g - (g_1 + \cdots + g_n)$

since $\varphi_j = 0$ outside $|z_j| = r_2$, supp g is compact

$\Rightarrow$ supp(g_0) is compact in $X' = X \setminus \{a_1, \ldots, a_n\}$

$$\Rightarrow \iint_X \frac{df}{f} \wedge dg_0 \overset{\text{property of } \wedge}{\underset{\text{defn of } X'}{=\!=}} -\iint_{X'} d\left(g_0 \frac{df}{f}\right) \overset{10.20}{=\!=} 0$$

(Recall 10.20: X: RS. $\omega \in \mathcal{E}^{(1)}(X)$ with compact support
Then $\iint_X d\omega = 0$)

~~Thus,~~ ~~$g_0 =$~~

Thus. $$\iint_X \frac{df}{f} \wedge dg \overset{\text{defn of } g_0}{\underset{\text{and what has been proved before}}{=\!=}} \iint_X \frac{df}{f} \wedge (g_1 + \cdots + g_n)$$

$$\underset{\text{supp}\, g_j \subseteq U_j}{=\!=} \sum_{j=1}^n \iint_{U_j} \frac{df}{f} \wedge dg_j$$

$$= \sum_{j=1}^n k_j \iint_{U_j} \frac{dz_j}{z_j} \wedge dg_j + \sum_{j=1}^n \iint_{U_j} \frac{d\psi_j}{\psi_j} \wedge dg_j$$

Thus $$\iint_{U_j} \frac{dz_j}{z_j} \wedge dg_j \overset{\text{property of } \wedge}{=\!=} -\lim_{\varepsilon \to 0} \iint_{\varepsilon \le |z_j| \le r_2} d\left(g_j \frac{dz_j}{z_j}\right)$$

$$\overset{\text{Stoke's}}{=\!=} -\lim_{\varepsilon \to 0} \iint_{\partial B_\varepsilon} d\left(g_j \frac{dz_j}{z_j}\right)$$

where $B_\varepsilon := \{\varepsilon \le |z_j| \le r_2\}$

$$\underset{g_j = 0 \text{ on } |z_j| = r_2}{=\!=} \lim_{\varepsilon \to 0} \int_{|z_j| = \varepsilon} g_j \frac{dz_j}{z_j}$$

$$\underset{\text{Cauchy integral formula}}{=\!=} 2\pi i\, g_j(a_j)$$

$$\underset{\text{By defn of } g_j = \psi_j g,\ \text{supp}\, \psi_j}{=\!=} 2\pi i\, g(a_j)$$

Similarly. $\iint_{U_j} \frac{d\psi_j}{\psi_j} \wedge dg_j = 0$, since $\psi_j(x) \neq 0$ $\forall x \in U_j$. ∎

④

20.4 Chains, Cycles and homology on RS X:

Def. 1-chain: $c = \sum_{j=1}^{k} n_j c_j$, $n_j \in \mathbb{Z}$.   Set of 1-chains: $C_1(X)$ is abelian group

$c_j : [0,1] \to X$ curves.

Def. $\forall\, \omega \in \mathcal{E}^{(1)}(X)$ closed,

$$\int_c \omega := \sum_{j=1}^{k} n_j \int_{c_j} \omega.$$

Def. boundary operator $\partial : C_1(X) \longrightarrow \mathrm{Div}(X)$

$c : [0,1] \to X$ curve

if $c(0) = c(1)$. define $\partial c := 0$

~~if~~ otherwise, $\partial c = \begin{cases} +1 & \text{at } c(1) \\ -1 & \text{at } c(0) \\ \text{otherwise } 0. \end{cases}$ $\boxed{\in \mathrm{Div}(X)}$

for $c = \sum n_j c_j$. define $\partial c := \sum n_j \partial c_j$

Rmk. $\forall\, c \in C_1(X)$. $\deg(\partial c) = 0$, by definition.

~~Claim~~: **Rmk**: Claim: on compact RS X

Given D with $\deg D = 0$.

Then, $\exists$ 1-chain c s.t. $\partial c = D$.

Pf: since X is compact, by defn of divisor

$\Rightarrow \{x \in X \mid D(x) \neq 0\}$ is a finite set

$\Rightarrow$ we can write D as $D = D_1 + \cdots + D_K$,

where each D_j takes $+1$ at some b_j. -1 at some a_j

and 0 otherwise.

Let c_j be a curve from a_j to b_j

Define $c := c_1 + \cdots + c_K$.

Then $\partial c = D$. by definition. ✓

Def. $Z_1(X) := \ker\left(C_1(X) \xrightarrow{\partial} \mathrm{Div}(X)\right)$

group of 1-cycles on X

Rmk. $\forall\, c$ closed curve. $\Rightarrow c \in Z_1(X)$ by definition.

Def. $c, c' \in Z_1(X)$ are homologous if

$\forall\, \omega \in \mathcal{E}^{(1)}(X)$ closed, $\int_c \omega = \int_{c'} \omega$.

The set of all homology classes of 1-cycles: $H_1(X)$.

called 1st homology group of X

It's an additive group, and abelian.

~~For r~~ Rmk. For $r \in H_1(X)$ and a closed $\omega \in \mathcal{E}^{(1)}(X)$,

$\int_r \omega$ is well-defined.

Rmk: By 10.10(a), two closed curves which are homotopic are also homologous.

Rmk: ~~That~~ Consider

$$\pi_1(X) \longrightarrow H_1(X)$$
$$[\sigma] \longmapsto [\sigma]$$

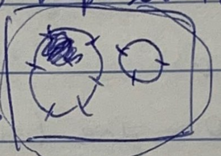
By 10.11, $\int_{\sigma\cdot\tau}\omega = \int_\sigma \omega + \int_\tau \omega$

then $[\sigma\cdot\tau] \longmapsto [\sigma]+[\tau]$

This is a group homomorphism.

Claim: it's surjective.

Pf: $\forall r \in H_1(X)$, $\exists c \in Z_1(X)$ representative of r (by defn of ∂c).

s.t. ~~c consists of~~ c is like this , since $\partial c = 0$.

Then, $\exists\, \sigma \in \pi_1(X)$ s.t. $\sigma \longmapsto r$. ✓

Claim: it's not in general injective.

Pf: since $\pi_1(X)$ is not in general abelian but $H_1(X)$ is abelian.

$\boxed{\text{20.5 Lemma}}$

X: RS. $c:[0,1]\to X$ curve.

U: relatively compact open nbhd of $c([0,1])$.

Then, $\exists$ a weak solution f of the divisor ∂c with $f|_{X\setminus U} = 1$,

s.t. $\forall$ closed $\omega \in \mathcal{E}^{(1)}(X)$,

$$\int_c \omega = \underbrace{\frac{1}{2\pi i}\iint_X \frac{df}{f}\wedge\omega}.$$

(note: $\frac{df}{f} = 0$ on $X\setminus U$, and $\bar{U}$ is compact

~~As by a Rmk before, the integral exists~~)

proof ①. consider the case: (U, z) is a coordinate nbhd on X s.t. $z(U) \subset \mathbb{C}$ is the unit disk and ~~the image~~

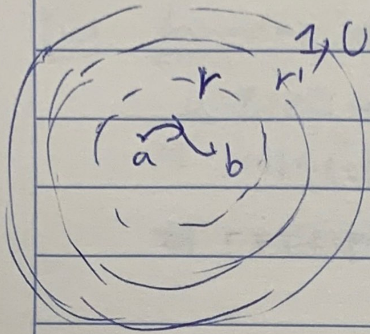
c lies entirely in U.

~~...~~. For simplicity, identify U with the unit disk.

$a := c(0)$. $b := c(1)$

$\exists\, r<1$ s.t. $c([0,1]) \subset \{|z|<r\}$

$\Rightarrow \log\left(\frac{z-b}{z-a}\right)$ has a well-defined branch in $\{r<|z|<1\}$



choose $\psi \in \mathcal{E}(U)$ with $\psi|_{\{|z|\le r\}} = 1$ and $\psi|_{\{|z|\ge r'\}} = 0$.

where $r<r'<1$.

Define $f_0 \in \mathcal{E}(U\setminus\{a\})$ by

$$f_0 = \begin{cases} \exp\left(\psi \log\frac{z-b}{z-a}\right) & r<|z|<1 \\ \frac{z-b}{z-a} & |z|\le r,\ z\neq a. \end{cases}$$

By $\psi|_{\{|z|\ge r'\}}=0$, we have $f_0|_{\{r'<|z|<1\}} = 1$.

we can continuously extend f_0 to $f \in \mathcal{E}(X\setminus\{a\})$ by defining it to be 1 on $X\setminus U$.

~~By calculating~~ Then f is a weak solution of ∂c.

$\forall\, \omega \in \mathcal{E}^{(1)}(X)$ closed.

By 10.4, ω has a primitive on U.

$\Rightarrow \exists\, g \in \mathcal{E}(X)$ with compact support s.t. $\omega = dg$ on $\{|z|\le r'\}$.

(note: outside this, $\frac{df}{f} = 0$, then don't need to consider ω during the integral).

Thus.

$$\frac{1}{2\pi i}\iint_X \frac{df}{f}\wedge\omega = \frac{1}{2\pi i}\iint_X \frac{df}{f}\wedge dg = g(b)-g(a)$$

20.13

note: $D = \partial c$

$$= \int_c dg = \int_c \omega.$$

②. In the general case,

$\exists$ a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$.

and coordinate nbhds (U_j, z_j), $j = 1, \ldots, n$ on X s.t.

$$\begin{cases} c([t_{j-1}, t_j]) \subset U_j \ \boxed{\subset U} \leftarrow \left(\text{note: } c([0,1]) \subset U\right) \\ z_j(U_j) \subset \mathbb{D} \quad \text{is the unit disk} \end{cases}$$

Let c_j denote $c([t_{j-1}, t_j])$.

By step ① construct f_j weak solution of divisor ∂c_j.

s.t. $f_j|_{X \setminus U_j} = 1$ and

$\forall \omega \in \mathcal{E}^{(1)}(X)$ closed, $\int_{c_j} \omega = \frac{1}{2\pi i} \iint_X \frac{df_j}{f_j} \wedge \omega$.

Then, define $f := f_1 \cdots f_n$.

then $\frac{df}{f} = \frac{df_1}{f_1} + \cdots + \frac{df_n}{f_n}$.

$f|_{X \setminus U} = 1$.

and ~~satisfy~~ $\forall \omega \in \mathcal{E}^{(1)}(X)$ closed. satisfying the required ~~equation~~ equality. $\blacksquare$

$\boxed{\text{20.6 Corollary}}$

X: compact RS.

Then ~~given~~ $\forall$ closed curve α on X.

$\exists$ a unique harmonic differential form $\sigma_\alpha \in \mathrm{Harm}^1(X)$

s.t. $\forall \omega \in \mathcal{E}^{(1)}(X)$ closed, $\int_\alpha \omega = \iint_X \sigma_\alpha \wedge \omega$

proof $\boxed{\text{existence}}$: for closed curve α. use ~~20.~~ 20.5.

$\Rightarrow \exists$ a weak solution of divisor $\partial\alpha = 0$

~~s.t. $f|_{X \setminus U} = 1$~~ satisfying conditions of 20.5.

$\frac{df}{f}$ is differentiable and closed on all of X. $\left(\text{since } d\left(\frac{1}{f} df\right) = -\frac{df}{f^2} \wedge df\right)$

$\left[\ \boxed{\begin{array}{l}\text{Recall 19.12: } X \text{ compact RS. Then} \\ \mathrm{Ker}\left(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X)\right) = d\mathcal{E}(X) \oplus \mathrm{Harm}^1(X)\end{array}}\ \right]$

Use 19.12, $\exists\ \sigma_\alpha \in \mathrm{Harm}^1(X)$, $g \in \mathcal{E}(X)$ s.t.

$$\frac{1}{2\pi i} \frac{df}{f} = \sigma_\alpha + dg.$$

$\forall \omega \in \mathcal{E}^{(1)}(X)$ closed $\Rightarrow dg \wedge \omega = d(g\omega)$.

$$\int_\alpha \omega \underset{20.5}{=} \frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge \omega = \iint_X \sigma_\alpha \wedge \omega + \left[\iint_X \underset{\substack{\| \\ d(g\omega)}}{dg \wedge \omega}\right] \overset{20.20}{=} 0. \text{ since } X \text{ is compact}$$

(then $g\omega$ has compact support)

Uniqueness.

Suppose $\sigma' \in \mathrm{Harm}^1(X)$ also ~~satis~~ satisfies this.

$\tau := \sigma_\alpha - \sigma'$

$\Rightarrow \boxed{\iint_X \tau \wedge \omega} = \iint_X (\sigma_\alpha - \sigma') \wedge \omega = \int_\alpha \omega - \int_\alpha \omega = \boxed{0}$,

$\forall \omega \in \mathcal{E}^{(1)}(X)$ closed.

Since τ is also $\in \mathrm{Harm}^1(X)$.

by def^n. $d * \tau = 0 \Rightarrow * \tau$ is closed.

$\Rightarrow \iint_X \tau \wedge * \tau = 0$.

$\Rightarrow$ ~~$\tau = \sigma_\alpha - \sigma'$~~ $\tau = 0$. ∎

20.7 Abel's Theorem

X: compact RS. D: divisor with $\deg D = 0$

Then, D has a solution

$\Longleftrightarrow \exists$ a 1-chain $c \in C_1(X)$ with

$\partial c = D$ s.t.

$\int_c \omega = 0$. $\forall \omega \in \Omega(X)$. ~~(defined as g)~~

Rmk: X compact. $\Rightarrow \dim \Omega(X) \underset{\text{Serre duality}}{=\!=} \dim H^1(X, \mathcal{O}) < \infty$ 14.10

$g := \dim H^1(X, \mathcal{O})$

$\Omega(X)$ is finite-dim vector space.

$\Rightarrow \int_c \omega = 0$ only has to be checked ~~on~~ for a basis of $\Omega(X)$

Rmk: Notice that

~~$\exists \alpha \in Z_1(X)$~~ $\exists \gamma \in C_1(X)$. s.t. $\partial \gamma = D$ ~~s.t.~~ $\exists \alpha \in Z_1(X)$ s.t $\int_\gamma \omega_j = \int_\alpha \omega_j$ $j = 1, \cdots, g$. where $\omega_1, \cdots, \omega_g$ is a basis of $\Omega(X)$	$\Rightarrow$	Let $c := \gamma - \alpha$ Then, $c \in C_1(X)$ $\partial c = D$ s.t. $\int_c \omega = 0$. $\forall \omega \in \Omega(X)$

proof of 20.17:

$\boxed{\Leftarrow}$: Suppose $c \in C_1(X)$ with $\partial c = D$ satisfying this.

~~By 20.15~~ ~~c = Σ finite sum of curves~~, $c = \sum^{\text{finite}} c_i$, do 20.15 to each c_i

~~Use 20.15~~ $\Rightarrow$ $\exists$ a weak solution f of D s.t. Define $f = f_1 \cdots f_m$

$\forall \omega \in \mathcal{E}^{(1)}(X)$ closed, $\int_c \omega = \frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge \omega$.

Then. $\forall \omega \in \Omega(X)$.

$$0 \underset{9.16}{=} \int_c \omega = \frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge \omega = \frac{1}{2\pi i} \iint_X \frac{d''f}{f} \wedge \omega$$

$\forall \omega \in \Omega(X)$, ω is closed. since $\omega \in \Omega(X)$. Locally $\omega = h\,dz$, so $\frac{d'f}{f} \wedge d\omega = 0$

Notice $\frac{d''f}{f} \in \mathcal{E}^{0,1}(X)$.

Recall 19.10: X compact R.S. $\sigma \in \mathcal{E}^{0,1}(X)$.
Then, $d''f = \sigma$ has a solution $f \in \mathcal{E}(X) \Leftrightarrow \iint_X \sigma \wedge \omega = 0$. $\forall \omega \in \Omega(X)$

Use 19.10, $\Rightarrow$ $\exists g \in \mathcal{E}(X)$ s.t. $\underbrace{d''g = \frac{d''f}{f}}$.

Define $F := e^{-g} f$.

Since $e^{-g} \in \mathcal{E}(X)$ never vanish. $\Rightarrow$ F is a weak solution of D.

$d''F = (d''e^{-g}) f + e^{-g} d''f = 0$

$\Rightarrow$ F is holomorphic on X_D $\Rightarrow$ F is a meromorphic ~~function~~ solution of D.

$\boxed{\Rightarrow}$: w.m.a. $D \neq 0$.

Let $f \in M(X)$ s.t. $(f) = D$

By 4.24 f defines an n-sheeted covering $f: X \to \mathbb{P}^1$ for some $n \geq 1$.

Use 1.15 $f \in M(X)$. for each pole p. $f(p) := \infty$.
Then $f: X \to \mathbb{P}^1$ is a holomorphic mapping.
4.20: X is compact. $\Rightarrow$ f is proper
4.24 $\Rightarrow$ $f: X \to \mathbb{P}^1$ is n-sheeted covering (regarding counting multiplicities)

By 4.23, 2.11, 4.5(a). set of branch points is closed & discrete.

X compact $\Rightarrow$ finite branch points.

Suppose $a_1, \cdots, a_r \in X$ are the branch points of f.

Define $Y := \mathbb{P}^1 \setminus \{f(a_1), \cdots, f(a_r)\}$

$\forall w \in \Omega(X)$. Define $\sigma = \text{Trace}(w)$ on $\mathbb{P}^1$:

$\forall y \in Y$, $\exists$ open nbhd V s.t. $f^{-1}(V)$ is the disjoint union of open sets $U_1, \cdots, U_n \subset X$.

and $f|U_\nu \to V$ are biholomorphic ;

Define $\varphi : V \to U_\nu$ be the inverse of $f|U_\nu \to V$.

Define $\text{Trace}(w)|_V := \varphi_1^* w + \cdots + \varphi_n^* w$.

~~It~~ If one carries out the same construction on an open nbhd V' of another point of Y. then on the intersection one gets the same differential form.

(see P48): this is an elementary symmetric function of w

$\Rightarrow$ we can holomorphically continue $\text{Trace}(w)$ ~~all of~~ from Y to all of $\mathbb{P}^1$.

By B.J. $H^1(\mathbb{P}^1, \mathcal{O}) = 0$

Use Serre duality $\Rightarrow \Omega(\mathbb{P}^1) = 0$

$\Rightarrow \text{Trace}(w) = 0$.

Let γ be a curve on $\mathbb{P}^1$ from ∞ to 0, with the possible exception of its end points lies entirely in Y.

$\Rightarrow$ $f^{-1}(\gamma)$ consists of n curves $c_1, \cdots, c_n$ joins the poles of f with zeros of f.

Define $c := c_1 + \cdots + c_n$.

$\partial c = D = (f)$.

$\forall w \in \Omega(X)$, $\int_c w \underset{\uparrow}{=} \int_\gamma \text{Trace}(w) = 0$.

Trace(w) locally is $\varphi_1^* w + \cdots + \varphi_n^* w$

$\varphi_i \circ \gamma = c_i$, P72: $\int_{c_i} w = \int_{\varphi_i \circ \gamma} w = \int_\gamma \varphi_i^* w$

$\blacksquare$

~~2018 Application~~

note: P4: $\mathbb{C}/\Gamma$ is compact.

20.8 Application to Doubly-Periodic Functions

$r_1, r_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.

$P := \{t_1 r_1 + t_2 r_2 \mid 0 \le t_1 < 1,\ 0 \le t_2 < 1\}$

Suppose zeros $a_1, \dots, a_n \in P$, poles $b_1, \dots, b_n \in P$ are prescribed, where each point appears as often as its multiplicity demands.

Then, $\exists$ a meromorphic function which is doubly-periodic w.r.t. $\Gamma = \mathbb{Z}r_1 + \mathbb{Z}r_2$ and has zeros $a_1, \dots, a_n$ and poles $b_1, \dots, b_n$ $\Big\}$ $(\star)$

$\iff \sum_{k=1}^{n}(a_k - b_k) \in \Gamma$.

proof: Let D be the divisor: $\begin{cases} \text{zeros} & +1 \\ \text{poles} & -1 \\ \text{otherwise} & 0 \end{cases}$ on $\mathbb{C}/\Gamma$.

Use Abel's Theorem: D has a solution

$\iff \exists c \in C_1(\mathbb{C}/\Gamma)$ s.t. $\partial c = D$. and take any $\{w_1, \dots, w_g\}$ basis of $\Omega(\mathbb{C}/\Gamma)$, $\int_c w_i = 0 .. \ \forall i = 1, \dots, g$.

D has a solution $\iff (\star)$.

Choose curves c_k from b_k to a_k in $\mathbb{C}$.

$\pi: \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$ canonical projection

$c := \pi \circ c_1 + \cdots + \pi \circ c_n \in C_1(\mathbb{C}/\Gamma)$.

$\partial c = D$.

17.13 $\Rightarrow$ genus of $\mathbb{C}/\Gamma$ is 1 $\xRightarrow{\text{Serre duality}}$ $\dim \Omega(\mathbb{C}/\Gamma) = 1$

By 10.14 $\Rightarrow$ w on X induced by dz on $\mathbb{C}$ is $\in \Omega(\mathbb{C}/\Gamma)$

so. this is a basis of $\Omega(\mathbb{C}/\Gamma)$

$$\int_c w \underset{\pi^* w = dz}{=} \sum_{k=1}^{n} \int_{c_k} dz = \sum_{k=1}^{n}(a_k - b_k)$$

So, if $\sum_{k=1}^{n}(a_k - b_k) \in \Gamma$, then D has a solution ✓.

~~If D has a solution, since $\mathbb{C}/\Gamma$ is compact. The curve c we defined above then must have~~

~~$\deg D = \sum_{k=1}^{n}(a_k - b_k) \in \Gamma$ ✓.~~

(12)

if D has a solution, the ~~curve~~ 1-chain c' in the proof of Abel's Theorem is a finite sum of curves $c_1', \dots, c_n'$ s.t. each of these joins ~~the~~ a pole of f with a zero of f, and they are the preimages of the curve γ in $\mathbb{P}^1$ from ∞ to 0. Hence, ~~by~~

in $\mathbb{C}/\Gamma$ $\quad 0 \underset{\substack{\text{use} \\ \text{since } w \in \Omega(X), \text{ Abel's Theorem}}}{=} \int_{c'} w \underset{\pi^* w = dz}{=} \sum (a_k' - b_k) \underset{\{a_k'\}, \{b_k'\} \text{ is a permutation of } \{a_k\}, \{b_k\}}{=} \sum (a_k - b_k)$

$\Rightarrow \sum (a_k - b_k) \in \Gamma$. ✓

■