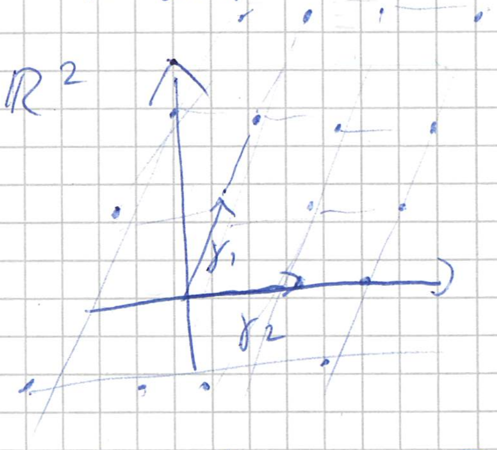
Seminar 22.5.

Introduction to lattices these will be needed to define the Jacobi variety. ~~[illegible]~~

Def Lattice: Suppose V is a N-dimensional VS. over $\mathbb{R}$. An additive subgroup is called a lattice if there exist N vectors $\gamma_1, \dots, \gamma_N \in V$, which are linearly independent over $\mathbb{R}$ s.t.

$$\Gamma = \mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_n$$

Picture intuition:



Theorem A subgroup $\Gamma \subset V$ is a lattice precisely if both of the following conditions hold:

i) Γ is discrete, (There exists a nbh. U of zero s.t. $\Gamma \cap U = \{0\}$)

ii) Γ is contained in in no proper vector subspace of V.

Compare with the picture for this case it obviously holds Now we want to proof it formaly

Omit proof, if enough time I'll show it.

Proof:

"$\Rightarrow$" A lattice clearly fulfills (i), (ii).

"$\Leftarrow$" We will prove this by induction on $N := \dim_{\mathbb{R}} V$

Suppose $\Gamma \subset V$ is a subgroup satisfying (i) and (ii).

For $N=0$ it holds trivially.

Induction step $N-1 \Rightarrow N$.
Since Γ is not contained in any proper vector subspace of V, there exists N linearly independent vectors $x_1, \dots, x_N \in \Gamma$.
Let V_1 be the vector subspace of V spanned by $x_1, \dots, x_{N-1}$ and let $\Gamma_1 := \Gamma \cap V_1$. Then the induction hypothesis may be applied to Γ_1. Thus there exist linearly independent vectors $\gamma_1, \dots, \gamma_{N-1} \in \Gamma_1 \subset \Gamma$ such that $\Gamma_1 = \mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_{N-1}$; Write $\Gamma \ni x = c_1(x)\gamma_1 + \dots + c_{N-1}(x)\gamma_{N-1} + c(x)x_N$
Consider the parallelotope $P := \{\lambda_1\gamma_1 + \dots + \lambda_{N-1}\gamma_{N-1} + \lambda x_N : \lambda_j, \lambda \in [0,1]\}$ it is compact. ~~That~~ Therefore $\Gamma \cap P$ is finite. Hence there exists a vector $\gamma_N \in (\Gamma \cap P) \setminus V_1$ such that $c(\gamma_N) = \min\{c(x) : x \in (\Gamma \cap P) \setminus V_1\} \in]0,1]$.
Claim $\Gamma = \Gamma_1 \oplus \mathbb{Z}\gamma_N$. Suppose $x \in \Gamma$ arbitrary. Then there exist $n_j \in \mathbb{Z}$ s.t. $x' := x - \sum_{j=1}^{N} n_j \gamma_j = \sum_{j=1}^{N-1} \lambda_j \gamma_j + \lambda x_N$.
where we have $0 \le \lambda_j < 1$ for $j = 1, \dots, N-1$. and $0 \le \lambda < c(\gamma_N)$. Since $x' \in \Gamma \cap P$, it follows from definition of the γ_N that $\lambda = 0$. Thus $x' \in \Gamma \cap V_1 = \Gamma_1$
Hence all λ_j are integers and thus are zero. $\Rightarrow x' = 0$
i.e. $x = \sum_{j=1}^{N} n_j \gamma_j \in \mathbb{Z}\gamma_1 + \dots \mathbb{Z}\gamma_N$. $\square$

Period Lattice Recall $\Omega(X)$ the vector space of holomorphic 1-forms on X.

Let X be a compact RS. of genus $g \geq 1$. and $\omega_1, \ldots \omega_g$ a basis of the VS $\Omega(X)$.

We define the subgroup $\mathrm{Per}(\omega_1, \ldots, \omega_g) \subset \mathbb{C}^g$ as follows $\mathrm{Per}(\omega_1, \ldots, \omega_g) := \{ (\int_\alpha \omega_1, \ldots, \int_\alpha \omega_g) \mid \alpha \in \pi_1(X) \}$

We will go on ~~to~~ to show that it is actually a lattice. It is called the period lattice of X relative to $\omega_1 \ldots \omega_g$.

For the proof that it is a lattice we need the following Lemma.

Suppose X is a compact RS of genus g. Then there are g distinct points $a_1 \ldots a_g \in X$ with the following property: Every holomorphic 1-from $\omega \in \Omega(X)$ which vanishes at all the points $a_1, \ldots, a_g$ is identically zero.

Note: it is just existence, it does not hold for all distinct g points.

Proof: For $a \in X$ let $H_a := \{ \omega \in \Omega(X) : \omega(a) = 0 \}$

Every H_a is either equal to $\Omega(X)$ or else has codimension 1, in $\Omega(X)$ (~~Blue~~ Hypeplane) $\bigcap_{a \in X} H_a = \{0\}$

and $\dim \Omega(X) = g$ $\Rightarrow$ There are g points $a_1, \ldots a_g \in X$ s.t $H_{a_1} \cap \ldots \cap H_{a_g} = 0$. These points satisfy the conditions of the Lemma $\square$.

First main theorem of todays talks.

Thm Suppose X is a compact Riemann surface of genus $g \geq 1$ and $\omega_1, \dots, \omega_g$ is a basis of $\Omega(X)$. Then $\Gamma := Per(\omega_1, \dots, \omega_g)$ is a Lattice in $\mathbb{C}^g$.

Proof: Chose $a_1 \dots a_g$ as in the previous lemma, and disjoint simply connected coordinate neighborhoods (U_j, z_j) of a_j with $z(a_j) = 0$

Let $\omega_i = \varphi_{ij} \, dz_j$ on U_j

$A := (\varphi_{ij}(a_j))$ has rank g. (By previous lemma,

(Suppose not $\to \begin{pmatrix} \quad \\ 0 \, 0 \, 0 \, 0 \, 0 \end{pmatrix}$ with change of basis $\Rightarrow \omega_g = 0$.)

Now define a mapping $F: U_1 \times \dots \times U_g \to \mathbb{C}^g$

$\underset{(x_1, \dots, x_g)}{\overset{}{x}} \mapsto (F_1(x), \dots, F_g(x))$ where

$$F_i(x) := \sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_i \qquad i = 1, \dots, g$$

Here the integral $\int_{a_j}^{x_j} \omega_j$ is along any curve from a_j to x_j which lies in U_j; since U_j is simply connected, the integral is independent of the curve chosen. The map F is complex differentiable with respect to the coordinates $z_1, \dots, z_g$ and has Jacobian matrix $J_F(x) = \left(\frac{\partial F_i}{\partial z_j}(x) \right) = \varphi_{ij}(x_j)$

Thus at $a = (a_1 \dots a_g)$ $J_F(a) = A$ is invertible. and

Hence $W := F(U_1 \times \dots \times U_g) \subset \mathbb{C}^g$ is a neighbourhood of $F(a) = 0$. (Version of the implicit function thm).

Next we will show $\Gamma \cap W = 0$.

Suppose $\exists \, \xi \in \Gamma \cap (W \setminus \{0\})$. Then there exists $x = (x_1, \dots, x_g) \in (U_1 \times \dots \times U_g)$ with $F(x) \in \Gamma$. Renumbering we may assume $x_j \neq a_j$ for $1 \leq j \leq k$ $x_j = a_j$ for $j > k$. where $1 \leq k \leq g$

By Abel's theorem. (last week)

there exists a meromorphic function f on X which has a pole of first order at a_j, and a zero of first order at x_j, $1 \le j \le k$, and is holomorphic otherwise.

Let $c_j z_j^{-1}$ be the principle part of f at a_j.

Of course $c_j \neq 0$ (~~meromor~~ meromorphic).

By the residue thm

$$0 = \text{Res}(f\omega_i) = \sum_{j=1}^{k} c_j \varphi_{ij}(a_j) \qquad \text{for } i = 1 \dots g.$$

This is impossible since rank $(\varphi_{ij}(a_j)) = g$

We have shown that Γ is a discrete subgroup of $\mathbb{C}^g$.

iii) Now we will show that Γ is not contained in any proper real vector subspace of $\mathbb{C}^g$. Otherwis there would exist a non-trivial real linear form on, which vanished identically on Γ. Since every real linear form is the real part of a complex linear form, one thus gets a vector $(c_1, \dots, c_g) \in \mathbb{C}^g / \{0\}$ such that $\text{Re}\left(\sum_{j=1}^{g} c_j \int_\alpha \omega_j\right) = 0 \quad \forall \alpha \in \pi_1(X)$

But by a previous Corollary (19.8)

$$\omega := c_1\omega_1 + \dots + c_g \omega_g = 0 \qquad \lightning$$

$\Rightarrow$ Γ is a lattice.

Cor 19.8 Suppose X is a compact RS and $\sigma \in \text{Harm}^1(X)$, $\omega \in \Omega(X)$. If for every closed curve γ on X one has

$$\int_\gamma \sigma = 0 \qquad \text{resp. } \text{Re}\left(\int_\gamma \omega\right) = 0.$$

then $\sigma = 0$ resp $\omega = 0$.