The Jacobi inverse problem

Let X be a compact RS of genus $g \geq 1$ and $\omega_1, \dots, \omega_g$ a basis of $\Omega(X)$. Set $\Gamma = \mathrm{Per}(\omega_1, \dots, \omega_g)$.

Basis exists since by Serre Duality: $g = \dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega) = \dim \Omega(X)$ on p 138, Remark 17.10. We saw that Γ is a lattice in $\mathbb{C}^g$.

D Jacobi variety: $\mathrm{Jac}(X) := \mathbb{C}^g / \Gamma$.

Since Γ is a lattice in $\mathbb{C}^g$, we know that $\mathrm{Jac}(X)$ is a complex g-torus. However we are now only interested in $\mathrm{Jac}(X)$ as an abelian group.

Further, the definition depends on the choice of $\omega_1, \dots, \omega_g$, but for another choice $\omega_1', \dots, \omega_g'$ we can easily get an isomorphism of groups $\mathbb{C}/\Gamma \cong \mathbb{C}/\mathrm{Per}(\omega_1', \dots, \omega_g')$.

D Picard group: $\mathrm{Pic}(X) := \mathrm{Div}(X) / \mathrm{Div}_p(X)$ and $\mathrm{Pic}_0(X) := \mathrm{Div}_0(X) / \mathrm{Div}_p(X)$

$\mathrm{Div}_0(X)$ are the divisors of degree 0 and $\mathrm{Div}_p(X)$ are the principal divisors, so divisors of the form (f) for $f \in \mathcal{M}(X) \setminus \{0\}$.

Note that $\deg(f) = 0$ since non-const. mero. func. on a compact RS have as many zeros as poles, counting multiplicity (p 80. Cor. 10.22).

Our goal will be to get a group isomorphism between $\mathrm{Jac}(X)$ and $\mathrm{Pic}_0(X)$.

R Recall $\overset{\text{chains}}{\partial : C_1(X)} \longrightarrow \mathrm{Div}(X)$, $c \longmapsto \partial c$, then $\ker \partial = \overset{\text{cycles}}{Z_1(X)}$ and we also get $\mathrm{im}\, \partial = \mathrm{Div}_0(X)$. Also $H_1(X) = Z_1(X) / \sim_H$

$\partial\big(\underbrace{\textstyle\sum_j n_j (a_j \to b_j)}_{=c}\big) = \sum_j n_j (D_{b_j} - D_{a_j})$ and $\deg(\partial c) = \sum_j n_j (1-1) = 0$.

Conversely, if $\deg D = 0 \Rightarrow D = \sum_j D_{b_j} - D_{a_j}$ and hence $\partial\big(\sum_j (a_j \to b_j)\big) = D$.

D $\Phi : \mathrm{Div}_0(X) \longrightarrow \mathrm{Jac}(X)$, $D = \partial c \longmapsto \left(\int_c \omega_1, \dots, \int_c \omega_g\right) + \Gamma$

This is well-defined since: $\partial c = D = \partial c' \Rightarrow \partial(c - c') = 0 \Rightarrow c - c' \in Z_1(X)$

and hence $\left(\int_c \omega_1, \dots, \int_c \omega_g\right) - \left(\int_{c'} \omega_1, \dots, \int_{c'} \omega_g\right) = \left(\int_{c-c'} \omega_1, \dots, \int_{c-c'} \omega_g\right) \in \Gamma$.

P $\ker \Phi = \mathrm{Div}_P(X)$.

$\supseteq$: Let $f \in M(X) \setminus \{0\}$ and $(f) \in \mathrm{Div}_P(X) \Rightarrow$ Abels thm: $\exists c \in C_1(X) : \partial c = (f)$

and $\int_c \omega = 0 \quad \forall \omega \in \Omega(X) \Rightarrow \Phi((f)) = \left(\int_c \omega_1, \dots, \int_c \omega_g\right) + \Gamma = (0, \dots, 0) + \Gamma$.

$\subseteq$: Let $D \in \ker \Phi$ and $c \in C_1(X) : \partial c = D$, then $\Phi(D) = \left(\int_c \omega_1, \dots, \int_c \omega_g\right) \in \Gamma$,

so there is an $e \in Z_1(X)$: $\left(\int_c \omega_1, \dots, \int_c \omega_g\right) = \left(\int_e \omega_1, \dots, \int_e \omega_g\right) \Rightarrow \int_{c-e} \omega_i = 0$

$\forall i$ and hence by Abel's thm we get an $f \in M(X)$ such that

$(f) = \partial(c - e) = \partial c - \partial e = \partial c$. Hence $(f) = \partial c = D$ and thus $D \in \mathrm{Div}_P(X)$. □

R Hence $j : \mathrm{Pic}_0(X) \to \mathrm{Jac}(X)$, $D + \ker \Phi \longmapsto \Phi(D)$ is inj. and the inverse Jacobi problem is to find out if j is surjective.

T $\mathrm{Im}\, \Phi = \mathrm{Jac}(X)$.

Recall the map $F : U_1 \times \dots \times U_g \longrightarrow \mathbb{C}^g$, $(x_1, \dots, x_g) \longmapsto \left(\int_c \omega_1, \dots, \int_c \omega_g\right)$ where

$c = \sum_j (a_j \to x_j)$ and $\mathrm{Im}\, F$ is an open nh. of $0 \in \mathbb{C}^g$.

So let $v + \Gamma \in \mathrm{Jac}(X)$ arbitrary, then for sufficently large N we get

$\frac{1}{N} v \in \mathrm{Im} F$, so there is $x = (x_1, \dots, x_g) \in U_1 \times \dots \times U_g$: $\frac{1}{N} v = F(x) = \left(\int_c \omega_1, \dots, \int_c \omega_g\right)$

for $c = \sum_j (a_j \to x_j)$ and hence $\left(\int_{Nc} \omega_1, \dots, \int_{Nc} \omega_g\right) = N \cdot \frac{1}{N} v = v$.

Note that $\deg(\partial Nc) = 0$, so $\partial Nc \in \mathrm{Div}_0(X)$ and $\Phi(\partial Nc) = v + \Gamma$. □

R This gives us the exact sequence

$$0 \longrightarrow \mathbb{C} \setminus \{0\} \xrightarrow{g} M(X) \setminus \{0\} \xrightarrow{h} \mathrm{Div}_0(X) \xrightarrow{\Phi} \mathrm{Jac}(X) \longrightarrow 0$$

- $g : z \longmapsto$ const. func. z $\Rightarrow$ $\ker g = ?$ $\quad \mathrm{Im}\, g =$ Const. func. on X
- $h : f \longmapsto (f)$ $\Rightarrow$ $\ker h =$ all mero. func. on X that have no zeros or poles

$=$ constant by Cor 2.8 p 11 : X cRS, $f : X \to \mathbb{C}$ holo. $\Rightarrow f$ const., so

$\ker h = \mathrm{Im}\, g$. $\quad \mathrm{Im}\, h = \mathrm{Div}_P(X)$.

D Let $a_1, \dots, a_g \in X$ be arbitrary : $J : X^g \to Jac(X)$, $x \mapsto (\int_c \omega_1, \dots, \int_c \omega_g) + \Gamma$

for $c = \sum_{j=1}^{g} (a_j \to x_j)$ and $x = (x_1, \dots, x_g)$.

This is w-d since if $c = \sum_{j=1}^{g} (a_j \to x_j)$ an $c' = \sum_{j=1}^{g} (a_j \to x_j)'$, then clearly

$\deg(\partial(c-c')) = 0$ and hence $(\int_c \omega_1, \dots, \int_c \omega_g) - (\int_{c'} \omega_1, \dots, \int_{c'} \omega_g) = (\int_{c-c'} \omega_1, \dots, \int_{c-c'} \omega_g) \in \Gamma$

because $c - c' \in Z_1(X)$.

T J is surjective.

Let $v + \Gamma \in Jac(X)$ arbitrary, then $\exists\, D \in Div_0(X) : \Phi(D) = v + \Gamma$.

Define $D' := D + D_{a_1} + \dots + D_{a_g}$, then $\deg D' = g$ and hence by Riemann-Roch

thm 16.9 on p129 we get $\dim H^0(X, \mathcal{O}_{D'}) \geq \dim H^0(X, \mathcal{O}_{D'}) - \dim H^1(X, \mathcal{O}_{D'})$

$= 1 - g + \deg D' = 1$, so there is an $f \in \mathcal{M}(X)$: $(f) \geq -D'$. Set

$D'' := D' + (f)$, then $D'' \geq 0$ and $\deg D'' = g$, so there are $x_1, \dots, x_g \in X$ such

that $D'' = D_{x_1} + \dots + D_{x_g}$. Hence $\sum_j D_{x_j} - D_{a_j} = D + (f)$.

Since $\ker \Phi = Div_P(X)$ we get $\Phi(\sum_j^g D_{x_j} - D_{a_j}) = \Phi(D + (f)) = \Phi(D) = v + \Gamma$

and if $c = \sum_j^g (a_j \to x_j)$ we get $J(c) = \Phi(\sum_j^g D_{x_j} - D_{a_j}) = v + \Gamma$.

$\square$

T If $g = 1$, then $J : X \to Jac(X)$ is an isomorphism.

If $g = 1$, then $J : x \mapsto \int_a^x \omega + Per(\omega)$ for some arbitrarily chosen

$a \in X$.

J is holo. : Let $c = \sum_{j=1}^{k} c_j$ for $c_j = (u_j \to u_{j+1})$ with c_j in (U_j, z_j) chart and

$u_1 = a$, $u_{k+1} = x$. Since $\omega \in \Omega(X)$, there is a primitive of ω on U_j, say $F_j \in \mathcal{O}(U_j)$

by Cor 10.7 p72. Hence $\int_c \omega = \sum_{j=1}^{k} \int_{c_j} \omega = \sum_{j=1}^{k} F_j(u_{j+1}) - F_j(u_j) = F_k(x) + t$

If we would have chosen another c, say c', then we would have gotten

$\int_{c'} \omega = F_k(x) + t'$ and we knew $\int_c \omega - \int_{c'} \omega \in \Gamma \Rightarrow t - t' \in \Gamma$. Hence

$J(x) = F_k(x) + t + \Gamma$ is holo. since F_k is holo.

J surj.: By the prev thm.

J inj.: Suppose J is not inj., then $J(x) = J(y)$ for $x \neq y$ in X. Hence

$\int_a^x w - \int_a^y w = \int_y^x w \in \Gamma$ and set $c = (x \to y)$. Then there is an $e \in Z_1(X)$

such that $\int_c w = \int_e w \Rightarrow \int_{c-e} w = 0$, so by Abels thm we get a $f \in \mathcal{M}(X)$

such that $(f) = \partial(c-e) = \partial c - \partial e = \partial c$. So f has a zero of order 1 at

y and a pole of order 1 at x. This is impossible since:

$f: X \to \mathbb{P}^1$ holo., non-const. and X compact $\Rightarrow$ f is surj. by Thm 2.7

on p11. f is also inj. since:

There is a thm that says $\sum_{x \in f^{-1}(a)} (f)(x)$ for $a \in \mathbb{P}^1$ is indep. of the choice

of a. If we choose $a = 0$, we see that the sum is equal to 1 since

f only has a root of order 1 at y. Hence for $a \neq 0$ we also get that

the sum is equal to 1 and since $(f) \geq 0$, this is only possible if

$|f^{-1}(a)| = 1$. Hence f is injective.

Now f is an isomorphism between X and $\mathbb{P}^1$, but X has genus 1 and $\mathbb{P}^1$ has genus 0 which is a contradiction. □

R. Every compact RS of genus 1 is isomorphic to a 1-torus by the previous theorem and 1-tori have genus 1 by Cor 17.13 on p 139, so up to isomorphism the only compact RS of genus 1 are 1-tori.

Question: If X has genus g and Y genus $k \neq g$, then does that mean they are not isomorphic (as Riemann surfaces)? If so, where does it say it in the book?