# 6 Sheaves

- In complex analysis one frequently deals with fn's which have various domains of definition.
- The notion of sheave is the suitable formal setting to handle this situation

## 6.1 Definition

Let $X$ be a top. space and $\mathcal{T}$ the topology (system of open sets)

A presheaf (of abelian groups) on $X$ is a pair $(\mathcal{F}, \rho)$ consisting of (or rings, vector spaces, sets etc.)

(i) a family $\mathcal{F} = (\mathcal{F}(U))_{U \in \mathcal{T}}$ of abelian groups

so we associate an abelian group to every open set in the topology.

(ii) a family $\rho = (\rho^U_V)_{U,V \in \mathcal{T},\, V \subset U}$ of group homomorphisms

$$\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \quad \text{where } V \text{ is open in } U$$

i.e. $V \cap U$ is open

with the following properties:

$$\rho^U_U = id_{\mathcal{F}(U)} \quad \forall\, U \in \mathcal{T}$$

$$\rho^V_W \circ \rho^U_V = \rho^U_W \quad \text{for } W \subset V \subset U$$

going from $\mathcal{F}(U)$ to $\mathcal{F}(V)$ then from $\mathcal{F}(V)$ to $\mathcal{F}(W)$ is the same as going from $\mathcal{F}(U)$ to $\mathcal{F}(W)$ directly, a sort of transitivity.

Remark The abelian groups can also be rings, vs's, sets etc.

Importantly, the homomorphisms $\rho^U_V$ are called restriction hom.'s

Instead of $\rho^U_V(f)$ one writes $f|V$.

**6.2.** Example Let $X$ be an arbitrary top. space.

$\forall\, U \subset X$ open let $\mathcal{C}(U) :=$ v.s. of cts fn's $f : U \to \mathbb{C}$.

For $V \subset U$ let $\rho^U_V : \mathcal{C}(U) \to \mathcal{C}(V)$ be the restriction map (precompose with inclusion map)

Then $\mathcal{C}$ is a presheaf of v.s's on $X$. notice one often drops the $\rho$ in the not. it's implied

Many examples of presheaves come from different classes of fn.

**6.3.** A presheaf is (called) a sheaf if $\forall\, U \subset X$ open and every open cover $U_i \subset U$, $i \in I$, s.t. $U = \bigcup_{i \in I} U_i$ the following conditions which we call the Sheaf Axioms hold:

(I) (locality) If $f, g \in \mathcal{F}(U)$ are elements s.t. $f|U_i = g|U_i$ $\forall\, i \in I$ then $f = g$.

(II) (gluing) Given elements $f_i \in \mathcal{F}(U_i)$ $i \in I$ s.t.

$$f_i|U_i \cap U_j = f_j| U_i \cap U_j \quad \forall\, i,j \in I$$

then $\exists\, f \in \mathcal{F}(U)$ s.t. $f|U_i = f_i$ $\forall\, i \in I$

Notice that by (I) this element $f$ is uniquely determined

Notice • The element $f$ whose $\exists$ is assured by (II) is uniquely determined by (I)

• special case: if $U = \emptyset$ then $\mathcal{F}(\emptyset)$ consists of one element (apply the SA) by (I) it's $\underset{\text{at most}}{\leq 1}$ by (II) it's $\underset{\text{at least}}{\geq 1}$

**6.4.** Examples

(a) $\forall$ top space $X$ the presheaf $\mathcal{C}$ in (6.2.) is a sheaf.
Given cts fn's $f_i : U_i \to \mathbb{C}$ which agree on the intersections check $\exists!$ cts fn $f : U \to \mathbb{C}$ whose restrictions equal the $f_i$
$! \to$ pointwise def $\quad \exists \to$ it's cts in an open set around every point

(b) On a Riemann Surface $X$ define the sheaf $\mathcal{O}$ of hol. fn's

by taking $\mathcal{O}(U)$ to be the ring of hol. fn's on $U \subset X$ open and the usual restriction maps. $\mathcal{O}(U) \to \mathcal{O}(V)$ for $V \subset U$
The sheaf $\mathcal{M}$ of mer. fn's is def'd analogously

(c) The sheaves $\mathcal{O}^*$ and $\mathcal{M}^*$ are defined on a R.S. by taking

$\mathcal{O}^*(U)$ to be the mult. group of hol. $f : U \to \mathbb{C}^*$

$\mathcal{M}^*(U)$ ——— mer. $f : U \to \mathbb{C}$

which do not vanish identically on any connected component of $U$. (You have to avoid zeroes as the inverse of poles)

+ restriction maps

(d) Now an example of a presheaf which is not a sheaf.

Let $X$ be a top space and $G$ abelian group.

$\forall\, U \subset X$ open define the constant presheaf

$$\mathcal{G}(U) = \begin{cases} 0 & \text{if } U = \emptyset \\ G & \text{otherwise} \end{cases}$$

The restriction mappings are

$$\rho^U_V = \begin{cases} 0 & \text{if } V = \emptyset \\ id_G & \text{otherwise} \end{cases}$$

If $G$ contains at least 2 distinct elements $g_1, g_2 \in G$ $g_1 \neq g_2$ and $X$ contains 2 disjoint non-empty open subsets $U_1, U_2$ then $\mathcal{G}$ is not a sheaf.

Why? Because condition (II) fails

$U_1 \cap U_2 = \emptyset$ so $g_1 | U_1 \cap U_2 = 0 = g_2 | U_1 \cap U_2$

but $\nexists\, f \in \mathcal{G}(U_1 \cap U_2) = G$ s.t. $f|U_1 = g_1$ & $f|U_2 = g_2$.

Let's modify this example to obtain a sheaf

(e) $\tilde{\mathcal{G}}(U)$ = locally cst. mappings $g: U \to G$ + usual restrictions.

Basically we assign a value to every connected component of $U$.

If $U$ is a non-empty connected open set $U \neq \emptyset$, open & connected

then this def. is equiv to the one above and $\tilde{\mathcal{G}}(U) = G$

Then $\tilde{\mathcal{G}}$ is called the sheaf of locally constant fn's

with values in $G$. Often denoted $G$.

Now we can define $f$ s.t. $f|U_1 = g_1$ & $f|U_2 = g_2$

since $U_1$ and $U_2$ are disjoint

## $\boxed{6.5.}$ The Stalk of a Presheaf (intuition: germs of $C^\infty$-fn's)

if you've heard about them in DG keep them in mind

Suppose $\mathcal{F}$ presheaf of sets on a top space $X$.

and $a \in X$ is a point. On the disjoint union

$$\dot{\bigcup_{U \ni a}} \mathcal{F}(U) \qquad U \text{ open nbhd of } a$$

remember, since $\mathcal{F}$ is a presheaf of sets the $\mathcal{F}(U)$ are sets

We introduce the equiv. rel. $\sim_a$ as follows:

$\underset{\mathcal{F}(U)}{\overset{}{\underset{\uparrow}{f}}} \overset{a}{\sim} \underset{\mathcal{F}(V)}{\underset{\uparrow}{g}}$ if $\exists W$ open s.t. $a \in W \subset U \cap V$ and $f|W = g|W$.

One can easily check this is an eq. rel. (Transitivity: intersection)'

The stalk of $\mathcal{F}$ at $a$ is the set $\mathcal{F}_a$ of all equiv. classes,

also called inductive limit of $\mathcal{F}(U)$.

$$\mathcal{F}_a := \varinjlim_{U \ni a} \mathcal{F}(U) := \left( \dot{\bigcup_{U \ni a}} \mathcal{F}(U) \right) / \overset{a}{\sim}$$

Q: What's the stalk of the constant presheaf?

$\mathcal{F}_a$ takes on the structure of the $\mathcal{F}(U)$'s, we define the operation on the equiv. classes by means of the op. def'd on the representatives. (This is indep. of the choice of rep.)

$\forall U$ nbhd of $a$, define $\rho_a : \mathcal{F}(U) \to \mathcal{F}_a$ which assigns to each element $f \in \mathcal{F}(U)$ its equiv. class modulo $\underset{a}{\sim}$.
$\rho_a(f)$ is called the germ of $f$ at $a$.

Example: germs of hol. (resp. mer.) fn's are Taylor (resp. Laurent) series. More precisely:

Let $X \subset \mathbb{C}$ be a domain, $a \in X$ and $\mathcal{O}$ the sheaf of hol. fn's
Given a germ, each of it's representatives is a hol. fn. in a nbhd of $a$ and thus has a Taylor expansion.
$\sum_{v=0}^{\infty} c_v (z-a)^v$ with +ve radius of conv.

Two hol. fn's on nbhds of $a$ represent the same germ precisely if they have the same Taylor expansion about $a$.
Thus the stalk $\mathcal{O}_a \cong \mathbb{C}\{z-a\}$ the ring of convergent power series in $z-a$ w complex coefficients
Similarly, $\mathcal{M}_a \cong$ the ring of convergent Laurent series

$$\sum_{v=k}^{+\infty} c_v (z-a)^v \quad k \in \mathbb{Z} \quad c_v \in \mathbb{C}$$

which have finite principal parts

Note: For any germ of a function $\varphi \in \mathcal{O}_a$, the value of the fn. $\varphi(a) \in \mathbb{C}$ is well-defined, i.e. indep. of the choice of rep.

6.6 Lemma: Suppose $\mathcal{F}$ sheaf of abelian groups on $X$ and $U \subset X$ open. Then an element $f \in \mathcal{F}(U)$ is zero iff all germs $\rho_x(f) \in \mathcal{F}_x$, $x \in U$ vanish.

Proof: This follows from Sheaf Axiom (I)
(bc we're saying that $\forall x$ there's an open nbhd around $x$ where $f$ is zero)

## 6.7. The Top. Space Associated to a Presheaf.

Suppose $X$ is a top. space and $\mathcal{F}$ a presheaf on $X$.
Let

$$|\mathcal{F}| := \dot{\bigcup_{x \in X}} \mathcal{F}_x$$

each one is the union of all the equiv. classes of germs at a.

be the disjoint union of all the stalks.

Define

$$p : |\mathcal{F}| \to X$$

which to each $\varphi \in \mathcal{F}_x$ assigns $x$. it returns the germ point

$\forall U \subset X$ open and $f \in \mathcal{F}(U)$ let

$$[U, f] := \{\rho_x(f) : x \in U\} \subset |\mathcal{F}|$$

for every point of $U$ we add the germ of $f$ at that pt to the set. We now prove this is a basis.

**6.8. Theorem** The set $\mathcal{B}$ of all $[U,f]$ s.t. $U \subset X$ open and $f \in \mathcal{F}(U)$ is a basis for a top. on $|\mathcal{F}|$. and the proj. $p : |\mathcal{F}| \to X$ is a local homeomorphism

Proof: (a) We have to check the following 2 properties

(i) The $[U,f]$ cover $|\mathcal{F}|$, which is trivial

(ii) If $\varphi \in [U,f] \cap [V,g]$ then $\exists [W,h] \in \mathcal{B}$ s.t. $\varphi \in [W,h] \subset [U,f] \cap [V,g]$.

For suppose $p(\varphi) = x$, then $x \in U \cap V$ and $\varphi = \rho_x(f) = \rho_x(g)$.

Hence $\exists W \subset U \cap V$ open nbhd of $x$ s.t. $f|W = g|W =: h$.

This implies $\varphi \in [W,h] \subset [U,f] \cap [V,g]$

(b) Now we show $p : |\mathcal{F}| \to X$ is a loc. hom.

Suppose $\varphi \in |\mathcal{F}|$ and $p(\varphi) = x$. $\exists \mathcal{B} \ni [U,f] \ni \varphi$. Then $[U,f]$ is an open nbhd of $\varphi$ and $U$ of $x$.

The map $p|[U,f] \to U$ is bijective and also cts and open as one sees from the def. Thus $p : |\mathcal{F}| \to X$ is a loc. homeo

**6.9.** **Def.** A presheaf $\mathcal{F}$ on $X$ satisfies the Identity Theorem if the following holds: Whenever $Y \subset X$ is a domain (non-empty, connected, open) and $f, g \in \mathcal{F}(Y)$ are elements s.t. their germs coincide at a pt $a$ $\rho_a(f) = \rho_a(g)$ with $a \in Y$, then $f = g$.

For example, this is true for the sheaves $\mathcal{O}$ and $\mathcal{M}$ of hol. and mer. fn's on a R.S. $X$.

Bc having the same germ corresponds to having the same Taylor/Laurent series around the germ point and that implies they're the same fn.

**6.10.** **Theorem:** Suppose $X$ loc. conn. Hausdorff space and $\mathcal{F}$ is a presheaf on $X$ which satisfies the Id thm. Then the top. space $|\mathcal{F}|$ is Hausdorff.

Proof: Suppose $\varphi_1, \varphi_2 \in |\mathcal{F}|$ and $\varphi_1 \neq \varphi_2$. We need to find disjoint nbhds of $\varphi_1$ and $\varphi_2$.

Case I: Suppose $p(\varphi_1) =: x \neq y := p(\varphi_2)$. Since $X$ is Hausdorff $\exists$ disjoint nbhds $U \ni x$ $V \ni y$. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint nbhds of $\varphi_1$ and $\varphi_2$ resp.

Case II: Suppose $p(\varphi_1) = p(\varphi_2) =: x$. Let the $\varphi_i \in \mathcal{F}_x$ be rep. by elements $f_i \in \mathcal{F}(U_i)$ where the $U_i$ are open nbhds of $x$.

Let $U \subset U_1 \cap U_2$ be a connected open nbhd of $x$. Then $[U, f_i|U]$ are open nbhds of $\varphi_i$. Now suppose $\exists \psi \in [U, f_1|U] \cap [U, f_2|U]$

Let $p(\psi) = y$. Then $\psi = \rho_y(f_1) = \rho_y(f_2)$. From the id. thm. it follows that $f_1|U = f_2|U$, thus $\varphi_1 = \varphi_2$ $\lightning$

Hence $[U, f_1|U]$ and $[U, f_2|U]$ are disjoint $\square$

# 9. Differential Forms

- Introduce the notion on Riemann surfaces
- Importantly: not only hol. & mer. forms but also forms which are only diff. in the real sense.

**9.1.** Suppose $U \subset \mathbb{C}$ open. Identifying $\mathbb{C} \sim \mathbb{R}^2$ by writing $z = x + iy$ where $x, y$ are the std real coords in $\mathbb{R}^2$. Write

$\mathcal{E}(U) :=$ $\mathbb{C}$-algebra of all those fn's $f: U \to \mathbb{C}$ which are $\infty$-diff wrt $x$ & $y$ (which is different from holomorphic fn's)

and define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

where $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are the usual partial derivatives

Note: Cauchy-Riemann $\Leftrightarrow$ $\mathcal{O}(U)$ is the kernel of $\frac{\partial}{\partial \bar{z}} : \mathcal{E}(U) \to \mathcal{E}(U)$

Check: Write $f = f_R + i f_I$ then

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f_R}{\partial x} - \frac{\partial f_I}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial f_I}{\partial x} + \frac{\partial f_R}{\partial y}\right) \Leftrightarrow \begin{cases} \frac{\partial f_R}{\partial x} = \frac{\partial f_I}{\partial y} \\ \frac{\partial f_R}{\partial y} = -\frac{\partial f_I}{\partial x} \end{cases}$$

(note taking the real and im. part of a fn commutes w/ the der.

**9.2** We can use complex charts to define the notion of diff'ble fn's on a RS

Def. Let $X$ be a RS and $Y \subset X$ open. Then $f: Y \to \mathbb{C}$ is $\infty$-diff'ble if $\forall$ chart $z: U \to V \subset \mathbb{C}$ on $X$ with $U \subset Y$ $\exists \tilde{f} \in \mathcal{E}(V)$ with $f|U = \tilde{f} \circ z$. Note: $\tilde{f} = f \circ z^{-1}$ is uniquely determined. The set of all such $f$ is called $\mathcal{E}(Y)$.

Together with the natural restr. map, we get the sheaf $\mathcal{E}$ of diff'ble fn's on the RS $X$.

diff'ble always means $\infty$-diff'ble

We can now define $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} : \mathcal{E}(U) \to \mathcal{E}(U)$ locally on RS's using complex charts $(U, z)$.

$\frac{\partial}{\partial z}\Big|_p f := \frac{\partial(f \circ z^{-1})}{\partial z}(z(p)) \overset{p = z^{-1}(z(p))}{\Longrightarrow} \frac{\partial f}{\partial z} \circ z^{-1} = \frac{\partial(f \circ z^{-1})}{\partial z}$

so $\frac{\partial f}{\partial z}$ is $C^\infty$ since the RHS and $z^{-1}$ are $C^\infty$

Let $a \in X$ and $\mathcal{E}_a$ the stalk of all the germs of diff'ble functions at $a$. Denote $m_a \subset \mathcal{E}_a$ the (vector) subspace of all fn germs which vanish at $a$ (remember the stalk inherits the structure from the type of sheaf by carrying out the operations on the representatives of the eq. class) and $m_a^2 \subset m_a$ the v.s. of those function germs $\varphi \in m_a$ which vanish to $2^{nd}$ order, i.e. if $\exists f$ repr. of $\varphi$ s.t. locally i.e. wrt a coord. nbhd $(U, z = x + iy)$ of $a$ one has

$$\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = 0 .$$

This def. is indep. of the choice of loc. coord. $z$. $\frac{\partial f}{\partial z} \circ w^{-1} = \underbrace{\frac{\partial f}{\partial z}}_{=0} \circ z^{-1} \circ z \circ w$

$\boxed{9.3}$ <u>Def</u> The quotient v.s.

$$T_a^{(1)} = \frac{m_a}{m_a^2}$$

is called the <u>cotangent space</u> of $X$ at the point $a$.

If $U$ is a open nbhd of $a$ and $f \in \mathcal{E}(U)$ then the differential $d_a f \in T_a^{(1)}$ of $f$ at $a$ is the element

$$d_a f := (f - f(a)) \bmod m_a^2$$

Note that $f - f(a)$ vanishes at $a$ so $\in m_a$. By definition its equiv. class mod $m_a^2$ is $d_a f$. intuition?

$\boxed{9.4}$ Theorem Suppose $X$ R.S., $a \in X$ and $(U, z = x+iy)$ a coord. nbhd. of $a$. Then $d_a x$ and $d_a y$ form a basis of the cotangent space $T_a^{(1)}$, as do $d_a z$ and $d_a \bar{z}$. If $f$ is a fn which is diff'ble on a nbhd of $a$, then

$$d_a f = \frac{\partial f}{\partial x}(a)\, d_a x + \frac{\partial f}{\partial y}(a)\, d_a y$$
$$= \frac{\partial f}{\partial z}(a)\, d_a z + \frac{\partial f}{\partial \bar{z}}\, d_a \bar{z}$$

Proof: (a) First show $d_a x, d_a y$ span $T_a^{(1)}$. Let $t \in T_a^{(1)}$ and suppose $\varphi \in m_a$ is a repr. of $t$. Taylor-expanding $\varphi$ about $a$ we get

$$\varphi = c_1(x - x(a)) + c_2(y - y(a)) + \psi$$

where $c_1, c_2 \in \mathbb{C}$ and $\psi \in m_a^2$. Taking both sides modulo $m_a^2$, we get

$$t = c_1 d_a x + c_2 d_a y$$

(b) Now we claim $d_a x$ and $d_a y$ are lin. ind.
If $c_1 d_a x + c_2 d_a y = 0$ then

$$c_1(x - x(a)) + c_2(y - y(a)) \in m_a^2$$

Then taking partial derivatives wrt $x, y$ we get $c_1 = c_2 = 0$

(c) Suppose $f$ is diff'ble in a nbhd of $a$. Then

$$f - f(a) = \frac{\partial f}{\partial x}(a)(x - x(a)) + \frac{\partial f}{\partial y}(a)(y - y(a)) + g$$

where $g \in m_a^2$. Thus

$$d_a f = \frac{\partial f}{\partial x}(a)\, d_a x + \frac{\partial f}{\partial y}(a)\, d_a y$$

The same steps work for $(d_a z, d_a \bar{z})$. $\square$

## 9.5 Cotangent vectors of type (1,0) and (0,1)

Suppose $(U,z)$ & $(U',z')$ are two coordinate nbhd's of $a \in X$. Then

$$\frac{\partial z'}{\partial z}(a) =: c \in \mathbb{C}^{*} \qquad ; \qquad \frac{\partial \bar{z}'}{\partial \bar{z}}(a) = \overline{\left(\frac{\partial z'}{\partial z}(a)\right)} = \bar{c}$$

bc $z \mapsto z'$ is invertible at a (in fact it's bihol.)

and

$$\frac{\partial z'}{\partial \bar{z}}(a) = \frac{\partial \bar{z}'}{\partial z}(a) = 0$$

$$\left[\ \frac{\partial \bar{z}'}{\partial \bar{z}}(a) = \frac{1}{2}\left(\frac{\partial \bar{z}'}{\partial x} + i\frac{\partial \bar{z}'}{\partial y}\right)\Big|_a = \frac{1}{2}\left(\frac{\partial x'}{\partial x} - i\frac{\partial y'}{\partial x} + i\frac{\partial x'}{\partial y} + \frac{\partial y'}{\partial y}\right)\Big|_a \qquad \frac{\partial z'}{\partial z}(a) = \frac{1}{2}\left(\frac{\partial z'}{\partial x} - i\frac{\partial z'}{\partial y}\right)\Big|_a = \frac{1}{2}\left(\frac{\partial x'}{\partial x} + i\frac{\partial y'}{\partial x} - i\frac{\partial x'}{\partial y} + \frac{\partial y'}{\partial y}\right)\Big|_a\ \right]$$

just plug values in

This implies that

$$d_a z' = \frac{\partial z'}{\partial z} d_a z + \cancel{\frac{\partial z'}{\partial \bar{z}}}^{\,0} d_a \bar{z} = c\, d_a z$$

and

$$d_a \bar{z}' = \cancel{\frac{\partial \bar{z}'}{\partial z}}^{\,0} d_a z + \frac{\partial \bar{z}'}{\partial \bar{z}} d_a \bar{z} = \bar{c}\, d_a \bar{z}$$

Thus the 1-dim vss's of $T_a^{(1)}$ which are spanned by $d_a z$ and $d_a \bar{z}$ are indep. of choice of local coords.

not true for x, y since we can scale one and not the other.

Write

$$T_a^{1,0} := \mathbb{C}\, d_a z \qquad T_a^{0,1} := \mathbb{C}\, d_a \bar{z}.$$

· By construction

$$T_a^{(1)} = T_a^{1,0} \oplus T_a^{0,1}$$

cotangent vectors of type (1,0) — " " " " (0,1)

If $f$ is diff'ble in a nbhd of $a$, define $d'_a f$ and $d''_a f$ by

$$d_a f = d'_a f + d''_a f \qquad d'_a f \in T_a^{1,0} \qquad d''_a f \in T_a^{0,1}$$

Then

$$d'_a f = \frac{\partial f}{\partial z}(a)\, d_a z \quad \text{and} \quad d''_a f = \frac{\partial f}{\partial \bar{z}}(a)\, d_a \bar{z}$$

(coordinates are unique by linear independence)

**9.6** **Def.** Suppose $X$ R.S. and $Y \subset X$ open.
A differential form of degree 1 or 1-form on $Y$ is a map

$$\omega : Y \longrightarrow \bigcup_{a \in Y} T_a^{(1)}$$

with $\omega(a) \in T_a^{(1)}$ $\forall a \in Y$. If $\omega(a) \in T_a^{1,0}$ (resp. $T_a^{0,1}$) $\forall a \in Y$ then $\omega$ is said to be of type $(1,0)$ (resp. $(0,1)$)

**9.7** Examples

(a) If $f \in \mathcal{E}(Y)$ then define the 1-forms $df, d'f, d''f$ by

$$(df)(a) := d_a f \quad (d'f)(a) := d'_a f \quad (d''f)(a) = d''_a f \quad \forall a \in Y.$$

Note: $f$ is hol. $\Leftrightarrow d''f = 0$ by the same reasoning as previously

(b) The pointwise product of a fn. and a 1-form is also a 1-form.

Remark: Locally (on a complex chart $(U, z = x+iy)$) every 1-form can be written

$$\omega = f\,dx + g\,dy = \varphi\,dz + \psi\,d\bar{z}$$

where the fn's $f, g, \varphi, \psi$ are not necessarily cts.

**9.8** **Def** Suppose $X$ R.S. and $Y \subset X$ open. A 1-form $\omega$ on $Y$ is called diff'ble (resp. hol.) if, wrt every chart $(U, z)$ we can write

$$\omega = f\,dz + g\,d\bar{z} \quad \text{on } U \cap Y \text{ where } f, g \in \mathcal{E}(U \cap Y)$$

resp.

$$\omega = f\,dz \quad \text{on } U \cap Y \text{ where } f \in \mathcal{O}(U \cap Y)$$

Notation $U \subset X$ open. $\mathcal{E}^1(U)$ = vs of diff'ble 1-forms
$\mathcal{E}^{1,0}(U)$ resp. $\mathcal{E}^{0,1}(U)$ the vss of type $(1,0)$ resp. $(0,1)$ forms
$\Omega(U)$ = vs of hol. 1-forms. These are all sheaves of vs's over $X$ (together with the usual restr. map.)

## 9.10. Meromorphic Differential Forms (skip if short on time)

def hol. except on a set of isolated poles

**9.10. Meromorphic Differential Forms.** A 1-form $\omega$ on an open subset $Y$ of a Riemann surface is said to be a meromorphic differential form on $Y$ if there exists an open subset $Y' \subset Y$ such that the following hold:

(i) $\omega$ is a holomorphic 1-form on $Y'$,
(ii) $Y \backslash Y'$ consists of only isolated points,
(iii) $\omega$ has a pole at every point $a \in Y \backslash Y'$.

$\mathcal{M}^{(1)}(Y)$ is a sheaf of v.s.'s over $X$.

## 9.9. The Residue

For $\omega$ as above, we can write $\omega = f dz$ on $Y'$ and def.
$\mathrm{Res}(\omega) = \mathrm{Res}(f)$ which is indep. of chart

## 9.11. The Exterior Product

**9.11. The Exterior Product.** In order to be able to define differential forms of degree two, we have to recall some properties of the exterior product of a vector space with itself. Let $V$ be a vector space over $\mathbb{C}$. Then $\Lambda^2 V$ is the vector space over $\mathbb{C}$ whose elements are finite sums of elements of the form $v_1 \wedge v_2$ for $v_1, v_2 \in V$. One has the following rules

$$(v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_2 \wedge v_3$$
$$(\lambda v_1) \wedge v_2 = \lambda(v_1 \wedge v_2)$$
$$v_1 \wedge v_2 = -v_2 \wedge v_1$$

for $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$. If $(e_1, \ldots, e_n)$ is a basis of $V$, then the elements $e_i \wedge e_j$, for $i < j$, form a basis of $\Lambda^2 V$. In fact these properties completely characterize $\Lambda^2 V$.

Set $T_a^{(2)} = \Lambda^2 T_a^{(1)}$

Locally (i.e. on a coordinate nbhd $(U, z = x+iy)$) $d_a x \wedge d_a y$ is a basis of $T_a^{(2)}$ and so is $d_a z \wedge d_a \bar{z} = -2i\, d_a x \wedge d_a y$

easy to check: write most general 2-form and simplify using wedge product properties.

Thus $T_a^{(2)}$ has dimension 1.

## 9.12. Def (2-form) just replace (1) by (2) in def of 1-form

Suppose $X$ R.S. and $Y \subset X$ open. A 2-form on $Y$ is a map

$$\omega : Y \to \bigcup_{a \in Y} T_a^{(2)}$$

with $\omega(a) \in T_a^{(2)}$ $\forall a \in Y$.

A 2-form $\omega$ is called differentiable on $Y \subset X$ open if wrt every complex chart $(U, z)$ on $X$ it can be written

$$\omega = f \, dz \wedge d\bar{z} \qquad \text{with } f \in \mathcal{E}(U \cap Y)$$

Denote $\mathcal{E}^{(2)}(Y)$ the vs of diff'ble 2-forms on $Y$.

There's no equivalent of holomorphic 1-forms for 2-forms

Examples: the (pointwise) wedge product of two 1-forms is a 2 form, and so is the pointwise product of a diff'ble fn and a 2-form.

## 9.13 Exterior Differentiation of Forms

We now define derivations $d, d', d'' : \mathcal{E}^{(1)}(U) \to \mathcal{E}^{(2)}(U)$ where $U$ is an open subset of a RS

Locally a diff'ble 1-form can be written as a finite sum

$$\omega = \sum f_k \, dg_k \qquad f_k, g_k \text{ diff'ble functions}$$

e.g. $\omega = f_1 dz + f_2 d\bar{z}$ where $z$ is a local coordinate

Set

$$d\omega := \sum df_k \wedge dg_k \quad d'\omega := \sum d'f_k \wedge dg_k \quad d''\omega := \sum d''f_k \wedge dg_k$$

The question is: is what I just wrote down well-defined?

Remains to show the def. is indep. of representation:

Suppose $\omega = \sum f_k \, dg_k = \sum \tilde{f}_j \, d\tilde{g}_j$. Choose a particular coord nbhd $(U, z = x + iy)$. wts that $\sum df_k \wedge dg_k = \sum d\tilde{f}_j \wedge d\tilde{g}_j$.

Because

$$dg_k = \frac{\partial g_k}{\partial x} dx + \frac{\partial g_k}{\partial y} dy$$

with a corresp. express. for $d\tilde{g}_j$ one has (by assumption)

$$\sum f_k \frac{\partial g_k}{\partial x} = \sum \tilde{f}_j \frac{\partial \tilde{g}_j}{\partial x}, \qquad \sum f_k \frac{\partial g_k}{\partial y} = \sum \tilde{f}_j \frac{\partial \tilde{g}_j}{\partial y}$$

Partially diff. wrt $x, y$ and subtracting we get

$$\sum \left( \frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} - \frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} \right) = \sum \left( \frac{\partial \tilde{f}_j}{\partial y} \frac{\partial \tilde{g}_j}{\partial x} - \frac{\partial \tilde{f}_j}{\partial x} \frac{\partial \tilde{g}_j}{\partial y} \right)$$

On the other hand

$$\sum df_k \wedge dg_k = \sum \left( \frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} - \frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} \right) dx \wedge dy$$

with a corresp. formula for $\sum d\tilde{f}_j \wedge d\tilde{g}_j$. The result follows.

$\boxed{9.14}$ Elementary properties

Let $f \in \mathcal{E}(U)$ and $\omega \in \mathcal{E}^{(1)}(U)$ as always $U$ is open sset of RS

Then

(i) $ddf = d'd'f = d''d''f = 0$

(ii) $d\omega = d'\omega + d''\omega$

(iii) $d(f\omega) = df \wedge \omega + f\, d\omega$

proof: (i) $ddf = d(1 \cdot df) = d1 \wedge df = 0$ similarly for $d', d''$

(ii) $d'\omega + d''\omega = \sum (d'f_k + d''f_k) \wedge dg$

$$= \sum \left( \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \right) \wedge dg = d\omega$$

(iii) $d(f\omega) = d \sum f f_k dg_k = \sum d(f f_k) \wedge dg_k$

$$= f \sum df_k \wedge dg_k + df \wedge \sum f_k dg_k = f d\omega + df \wedge \omega$$

Harmonic functions

From (i) and (ii) we get

$$d'd''f = -d''d'f \qquad \text{just expand } (d'+d'')^2$$

With respect to a local chart $(U, z = x+iy)$

$$d'd''f = \frac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \frac{1}{2i} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy \quad \text{(easy algebra)}$$

Hence a diff'ble fn $f$ on an open subset of a RS is called harmonic if $d'd''f = 0$

$\boxed{9.15.}$ $\omega \in \mathcal{E}^1(Y)$ is closed if $d\omega = 0$

exact if $\omega = df$ for $f \in \mathcal{E}(Y)$

Note exact $\Rightarrow$ closed since $ddf = 0$. The converse is false in general.

[9.16.] **Theorem** On an open subset $Y$ of a RS:

(a) Every holomorphic 1-form $\omega \in \Omega(Y)$ is closed

What follows is the converse since $\Omega \subset \mathcal{E}^{1,0}$

(b) Every closed 1-form $\omega \in \mathcal{E}^{1,0}(Y)$ is holomorphic

being of type 1,0 is clearly a necessary condition

*Proof*: Suppose $\omega \in \mathcal{E}^{1,0}(Y)$. Then we can locally write

$\omega = f\,dz$ for $f \in \mathcal{E}(Y)$ diff'ble

Then

by antisymmetry

$$d\omega = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) \wedge dz = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$$

Hence

$$d\omega = 0 \iff \frac{\partial f}{\partial \bar{z}} = 0$$ and the result follows □

Corollary If $u$ is a harmonic fn then $d'u$ is a hol. 1-form, since $dd'u = (d' + d'')d'u = d''d'u = 0$

[9.17] The pull-back of differential forms

Suppose $F: X \to Y$ is a hol. map between 2 RSs

$\forall\, U \subset Y$ open $F$ induces a homomorphism

$$F^*: \mathcal{E}(U) \to \mathcal{E}(F^{-1}(U))$$
$$f \mapsto f \circ F$$

generalising this to differential forms:

$$F^*: \mathcal{E}^{(k)}(U) \longrightarrow \mathcal{E}^{(k)}(F^{-1}(U)) \qquad k = 1,2$$

defined how?

Write the 1-form locally as the finite sum $\sum f_i\,dg_i$

2-form ——————————— $\sum f_i\, dg_j \wedge dh_j$

where $f_i, g_i, h_j$ are differentiable. Set

$$F^*\left(\sum f_j\, dg_j\right) = \sum (F^* f_j)\, d(F^* g_j)$$

$$F^*\left(\sum f_j\, dg_j \wedge dh_j\right) = \sum (F^* f_j)\, d(F^* g_j) \wedge d(F^* h_j)$$

it's easy to check these def's are indep of the local rep. chosen

and hence piece together to give unique global vs hom.

For $f \in \mathcal{E}(U)$ and $\omega \in \mathcal{E}^{(1)}(U)$ one has

$$F^*(df) = d(F^* f), \qquad F^*(d\omega) = d(F^*\omega)$$

and similarly with $d \to d', d''$.

<u>Consequence</u> if $f \in \mathcal{E}(U)$ is harmonic then $F^* f = f \circ F \in \mathcal{E}(F^{-1}(U))$ is also harmonic. For

$$d'd''(F^* f) = d'(F^* d'' f) = F^*(d'd'' f) = 0$$