Integration of differential ~~1~~ forms

10.1

X R.s., $\omega \in \mathcal{E}^{(1)}(X)$, $c : [0,1] \to X$ (piecewise continuously differentiable curve), $\exists\ 0 = t_0 < t_1 < \dots < t_n = 1$

(U_k, z_k), $z_k = x_k + i y_k$ charts s.t.

$c([t_{k-1}, t_k]) \subset U_k$ and

$x_k \circ c : [t_{k-1}, t_k] \to \mathbb{R}$, $y_k \circ c : [t_{k-1}, t_k] \to \mathbb{R}$

have continuous first order derivatives

On U_k: $\omega = f_k\, dx_k + g_k\, dy_k$

f_k, g_k differentiable the integral of ω along the curve c is defined as

$$\int_c \omega := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(f_k(c(t)) \frac{d x_k(c(t))}{dt} + g_k(c(t)) \frac{d y_k(c(t))}{dt} \right) dt$$

– Independent of choice of charts

Thm 10.2

X R.s. $c : [0,1] \to X$ piece-wise continuously differentiable curve and $F \in \mathcal{E}(X)$ then

$$\int_c dF = F(c(1)) - F(c(0))$$

proof: $0 = t_0 < t_1 < \dots < t_n = 1$, charts (U_k, z_k) as above.

On U_k one has (Thm 9.4)

$$dF = \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y_k}$$

Thus,

$$\int_c dF = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{\partial F}{\partial x_k}(c(t)) \frac{d x_k(c(t))}{dt} + \frac{\partial F}{\partial y_k}(c(t)) \frac{d y_k(c(t))}{dt} \right) dt$$

$$= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{d}{dt} F(c(t)) \right) dt$$

$$= \sum \left(F(c(t_k)) - F(c(t_{k-1})) \right) = F(c(1)) - F(c(0))$$ $\square$

Def 10.3

X R.S., $\omega \in \mathcal{E}^{(1)}(X)$

$F \in \mathcal{E}(X)$ is called primitive of ω if $dF = \omega$

Rmrk: (9.15) Any ~~primiti~~ form with primitive is closed (primitive is not unique)

10.4 local existence of primitives

Let $U := \{z \in \mathbb{C} : |z| < r\}$, $\omega \in \mathcal{E}^{(1)}(U) \rightarrow$

$\omega = f\,dx + g\,dy \quad f, g \in \mathcal{E}(U)$

Assume ω is closed ($d\omega = 0$) $\rightarrow$

$0 = d\omega = df \wedge dx + dg \wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$

(By Thm 9.4)

$\Leftrightarrow \quad \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$

Claim: ω has primitive F which is given by

$F(x,y) := \int_0^1 (f(tx,ty)x + g(tx,ty)y)\,dt$ for $(x,y) \in U$

$F \in \mathcal{E}(U)$ clear, only need to check $dF = \omega$

i.e. $(\partial F / \partial x) = f$, $(\partial F / \partial y) = g$:

$$\frac{\partial F(x,y)}{\partial x} = \int_0^1 \frac{\partial f}{\partial x}(tx,ty)\,tx + \frac{\partial g}{\partial x}(tx,ty)\,ty + f(tx,ty))\,dt$$

Since $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ and $\frac{d}{dt} f(tx,ty) = \frac{\partial f}{\partial x}(tx,ty)x + \frac{\partial f}{\partial y}(tx,ty)y$

$$\Rightarrow \frac{\partial F(x,y)}{\partial x} = \int_0^1 \left(t \frac{d}{dt} f(tx,ty) + f(tx,ty)\right) dt = f(x,y)$$

Similarly $(\partial F/\partial y) = g \Rightarrow dF = \omega$

If ω is holomorphic i.e. $\omega = f\,dz$ with $f \in \mathcal{O}(U)$

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be the taylor series expansion of f

$\rightarrow F(z) := \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1} \in \mathcal{O}(U)$ s.t. $dF = \omega$

Thm 10.5

X R.S., $\omega \in \mathcal{E}^{(1)}(X)$ closed. Then $\exists$ covering map $p: \hat{X} \to X$ with $\hat{X}$ connected, and a primitive $F \in \mathcal{E}(\hat{X})$ of the differential $p^*\omega$

proof:

Let $\mathcal{F}$ be the sheaf of primitives of ω:
$U \subset X$ open let $\mathcal{F}(U)$ consist of all fct's $f \in \mathcal{E}(U)$ s.t. $df = \omega$

Consider associated space $|\mathcal{F}| := \bigcup_{x \in X} \mathcal{F}_x$:

$p: |\mathcal{F}| \to X$, $\mathcal{F}_x \ni \varphi \mapsto x$

(6.10 follows $|\mathcal{F}|$ is hausdorff)

$p: |\mathcal{F}| \to X$ is a covering: For every pt $a \in X$ by 10.4 $\exists$ connected open nbhd U and a primitive $f \in \mathcal{F}(U)$ of ω. Then $f + c$, for $c \in \mathbb{C}$ are all primitives of ω on U. Hence

$$p^{-1}(U) = \bigcup_{c \in \mathbb{C}} [U, f+c]$$

The sets $[U, f+c]$ are pairwise disjoint and all the mappings $p|[U, f+c] \to U$ are homeos, thus $p: |\mathcal{F}| \to X$ is a covering map.

Let $\hat{X} \subset |\mathcal{F}|$ be a connected component $\to$ $p|\hat{X} \to X$ also a covering map

Let $F: \hat{X} \to \mathbb{C}$ be defined by $F(\varphi) := \varphi(p(\varphi))$

It then follows that F is a primitive of $p^*\omega$ $\blacksquare$

10.6. Corollary

X R.S. $\pi: \tilde{X} \to X$ universal covering, $\omega \in \mathcal{E}^{(1)}(X)$ closed then $\exists$ primitive $f \in \mathcal{E}(\tilde{X})$ of $\pi^*\omega$

proof: Let $p: \hat{X} \to X$ be the covering as in 10.5 and Let $F \in \mathcal{E}(\hat{X})$ be a primitive of $p^*\omega$

universal property $\Rightarrow$ $\exists$ holomorphic fiber preserving mapping $\tau: \tilde{X} \to \hat{X}$. Let $f := \tau^* F \in \mathcal{E}(\tilde{X})$ is a primitive of $\tau^*(p^*\omega) = \pi^*\omega$ $\boxed{\equiv}$

10.7 Corollary

Let X be a simply connected R.S. then every closed differential form ~~has~~ $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive $F \in \mathcal{E}(X)$

proof: Follows from 10.6. since $\mathrm{Id}: X \to X$ is the universal covering

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10.8 Thm

X R.S. $p: \hat{X} \to X$ universal covering. $\omega \in \mathcal{E}^{(1)}(X)$ closed, $F \in \mathcal{E}(\hat{X})$ primitive of $p^*\omega$. If $c: [0,1] \to X$ is piece-wise continuously differentiable and $\hat{c}: [0,1] \to \hat{X}$ lifting of c then

$$\int_c \omega = F(\hat{c}(1)) - F(\hat{c}(0))$$

proof:

$\forall\, v: [0,1] \to \hat{X}$ piece-wise continuously diff., $\forall\, \omega \in \mathcal{E}^{(1)}(X)$:

$$\int_v p^*\omega = \int_{p \circ v} \omega$$

~~This follows from def~~ By definition. Theorem follows from 10.2

$\blacksquare$

Rmk 10.9

10.8 gives way to define integral along arbitrary (continuous) curves $c: [0,1] \to X$.

This definition is independent of the choice of F of $p^*\omega$ as any two primitives only differ by constant.

Also independent of the lifting of curve

Thm 10.10

X R.S., $\omega \in \mathcal{E}^{(1)}(X)$ closed

a) If $a, b \in X$, $u, v: [0,1] \to X$ two homotopic curves from a to b then

$$\int_u \omega = \int_v \omega$$

b) If $u, v: [0,1] \to X$ are two closed curves which are free homotopic, then

$$\int_u \omega = \int_v \omega$$

(free homotopic as closed curves: start and endpoint are never separated during homotopy)

proof:

a) Let $p: \tilde{X} \to X$ be the universal covering and supp. $\hat{u}, \hat{v}: [0,1] \to \hat{X}$ are liftings of u and v resp. with same initial point.

Thm 4.10 implies that $\hat{u}, \hat{v}$ have same endpoint

Result follows from Thm 10.8

b) Supp. u has initial- and endpoint x_0 and v has initial- and endpoint x_1. Then $\exists$ curve w from x_0 to x_1 s.t. u is homotopic to $w \cdot v \cdot w^-$ hence by a):

$$\int_u \omega = \int_{w \cdot v \cdot w^-} \omega = \int_w \omega + \int_v \omega - \int_w \omega = \int_v \omega \quad \boxtimes$$

10.11 Periods

X R.S. $\omega \in \mathcal{E}^{(1)}(X)$ closed. By 10.10 one can define

$$a_\sigma := \int_\sigma \omega \quad , \quad \sigma \in \pi_1(X)$$

by choosing representative curve of hom. class σ and integrating.

These integrals are called ~~primitives~~ periods of ω

One has

$$\int_{\sigma \cdot \tau} \omega = \int_\sigma \omega + \int_\tau \omega \quad \text{for } \sigma, \tau \in \pi_1(X) \quad \to$$

Induces homom. $\pi_1(X) \to \mathbb{C}$ which is called period homom. associated to the closed differential form ω

Ex.:

Supp. $X = \mathbb{C}^*$ then $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$

generator of $\pi_1(\mathbb{C}^*)$: $u : [0,1] \to \mathbb{C}^*$, $t \mapsto e^{2\pi i t}$

Let $\omega := (dz/z)$ where z is canonical coordinate

then

$$\int_u \omega = \int_u \frac{dz}{z} = 2\pi i$$

Hence period homo of ω is

$\mathbb{Z} \to \mathbb{C}$, $n \mapsto 2\pi i n$

10.12 Summands of Automorphy

X R.S., $p : \tilde{X} \to X$ universal covering

The group $G := \mathrm{Deck}(\tilde{X}/X)$ of covering transformations

is isomorphic to $\pi_1(X)$

If $\sigma \in G$, $f : \tilde{X} \to \mathbb{C}$ we can define

$\sigma f : \tilde{X} \to \mathbb{C}$ by $\sigma f := f \circ \sigma^{-1}$.

If $g : \tilde{X} \to \mathbb{C}$ is other function:

$\sigma(f+g) = \sigma f + \sigma g$, $\sigma(fg) = (\sigma f)(\sigma g)$

Also for $\sigma, \tau \in G$: $(\sigma\tau) f = \sigma(\tau f)$

$f : \tilde{X} \to \mathbb{C}$ is called

additively automorphic with const. summands of automorphy if

$\exists\, a_\sigma \in \mathbb{C}$, $\sigma \in G$ s.t.

$f - \sigma f = a_\sigma \quad \forall\, \sigma \in G$

a_σ are called summands of automorphy of f

10.13

X Riemann surface $p: \tilde{X} \to X$ universal covering

a) $\omega \in \mathcal{E}^{(1)}(X)$ closed differential form, $F \in \mathcal{E}(\tilde{X})$ primitive of $p^*\omega$ then

F is additively automorphic with const. summands of automorphy

b) Supp. $F \in \mathcal{E}(\tilde{X})$ additive automorphic with const summands of automorphy then

$\exists! \, \omega \in \mathcal{E}^{(1)}(X) : dF = p^*\omega$

proof:

a) For $\sigma \in G$: $p \circ \sigma^{-1} = p$ thus σF is also primitive of $p^*\omega \Rightarrow$

$-a_\sigma := \sigma F - F$ is constant

Supp. $x_0 \in X$ and $z_0 \in \tilde{X}$ with $p(z_0) = x_0$. Supp. $\sigma \in \mathrm{Deck}(\tilde{X}/X)$

By 5.6. $\bar{\sigma} \in \pi_1(X, x_0)$ can be represented as follows:

Choose curve $v: [0,1] \to \tilde{X}$ with $v(0) := y_0 := \sigma^{-1}(z_0)$ and $v(1) := z_0 = \sigma(y_0) \Rightarrow$

$u := p \circ v$ is closed curve in X, $\bar{\sigma} = cl(u)$

10.8 $\Rightarrow$ periods of ω wrt σ are given by

$$\int_u \omega = F(v(1)) - F(v(0)) = F(z_0) - F(\sigma^{-1}(z_0)) = -a_\sigma$$

b) Let $a_\sigma \in \mathbb{C}$ be the summands of automorphy, then $\forall \, \sigma \in \mathrm{Deck}(\tilde{X}/X)$:

$\sigma^*(dF) = d\sigma^*F = d(F + a_\sigma) = dF$

$\Rightarrow dF$ is inv. under covering transformations

Since $p: \tilde{X} \to X$ is locally bihol. $\exists \, \omega \in \mathcal{E}^{(1)}(X)$: $dF = p^*\omega$. ω is uniquely determined, and closed.

10.15

X Riemann surface. $\omega \in \mathcal{E}^{(1)}(X)$ closed then ω has primitive $f \in \mathcal{E}(X)$ iff all periods of ω are zero.

proof:

"$\Rightarrow$" 10.2

"$\Leftarrow$" By 10.6 $\exists$ primitive $F \in \mathcal{E}(\tilde{X})$ of $p^*\omega$ for the universal covering $p: \tilde{X} \to X$

By 10.3 F has summ. of autom. 0. Thus, $\exists f \in \mathcal{E}(X)$ s.t. $F = p^* f$. f is primitive of ω since

$$p^*\omega = dF = d(p^* f) = p^*(df) \rightsquigarrow \omega = df$$ $\square$

Rmk: All periods of $\omega = 0$ one gets a primitive of ω from the integral

$$f(x) := \int_{x_0}^{x} \omega$$

where $x_0 \in X$ is arbitrary and integral is along any curve from x_0 to x

Differential 2-Forms

10.17. suppose $U \subset \mathbb{C}$ open, $\omega \in \mathcal{E}^{(2)}(U) \rightsquigarrow$

$$\omega = f\, dx \wedge dy = \frac{i}{2} f\, dz \wedge d\bar{z}, \quad f \in \mathcal{E}(U)$$

Assume ~~$\S$~~ $\mathrm{supp}(f) \subset U \rightsquigarrow$ define

$$\iint_U \omega := \iint_U f(x,y)\, dx\, dy$$

For $\varphi: V \to U$ biholomorphic mapping
$V \subset \mathbb{C}$

$\varphi = u + iv$

Jacobian determinante is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = |\varphi'|^2 \rightsquigarrow$$

$$\iint_U f\, dx\, dy = \iint_V (f \circ \varphi)|\varphi'|^2\, dx\, dy$$

Furthermore

$\varphi^*(dz \wedge d\bar{z}) = d\varphi \wedge d\bar{\varphi} = (\varphi' dz \wedge \bar{\varphi}' d\bar{z}) = |\varphi'|^2 dz \wedge d\bar{z}$

$\rightsquigarrow \varphi^* \omega = (f \circ \varphi) |\varphi'|^2 dx \wedge dy$ Hence

$$\iint_U \omega = \iint_V \varphi^* \omega$$

10.18

X Riemann surface

a) $\varphi : U \subset X \to V \subset \mathbb{C}$ chart, $\omega \in \mathcal{E}^{(2)}(X)$ s.t. $\mathrm{supp}(\omega) \Subset U$ $\rightsquigarrow$

$$\iint_X \omega := \iint_U \omega := \iint_V (\varphi^{-1})^* \omega$$

$\rightsquigarrow$ independent of choice of chart

b) $\omega \in \mathcal{E}^{(2)}(X)$ with compact support:

$\mathrm{supp}(\omega) \subset \bigcup_{k=1}^{n} U_k$

$\varphi_k : U_k \to V_k$ charts

$f_k \in \mathcal{E}(X)$ partition of unity s.t.

$\mathrm{supp}(f_k) \subset U_k$ $\rightsquigarrow$

$\omega = \sum f_k \omega$ define

$$\iint_X \omega := \sum_{k=1}^{n} \iint_X f_k \omega$$

where rhs is defined as in a)

10.19 Stokes thm in the plane

$U \subset \mathbb{C}$ open, $A \subset U$ compact with smooth ∂A:

$$\iint_A d\omega = \int_{\partial A} \omega \quad \forall \omega \in \mathcal{E}^{(1)}(U)$$

10.20

X Riemann surface, $\omega \in \mathcal{E}^{(1)}(X)$ with compact support then

$$\iint_X d\omega = 0$$

proof

By multiplying with partition of unity $\rightarrow$

$\omega = \omega_1 + \dots + \omega_n$ s.t. ω_k has compact support in a chart

W.l.o.g. $X = \mathbb{C}$

Let $R > 0$: $\mathrm{supp}(\omega) \in \{z \in \mathbb{C} : |z| < R\}$ then By stokes

$$\iint_{\mathbb{C}} d\omega = \iint_{|z| \leq R} d\omega = \int_{|z|=R} \omega = 0$$

10.21 Residue thm

X compact Riemann surface $a_1, \dots, a_n \in X$

$X' := X \setminus \{a_1, \dots, a_n\}$. Then for all $\omega \in \Omega(X')$:

$$\sum_{k=1}^{n} \mathrm{Res}_{a_k}(\omega) = 0$$

proof:

choose coordinate nbhd's (U_k, z_k) of the a_k s.t. $U_j \cap U_k = \emptyset$ if $j \neq k$. We can assume $z_k(a_k) = 0$, $z_k(U_k) \subset \mathbb{C}$ is a disk

~~Let f_k~~ Choose $f_k : X \to \mathbb{C}$ s.t. $\mathrm{supp}(f_k) \subset U_k$ and $f_k|U_k' = 1$ for an open nbhd $U_k' \subset U_k$ of a_k

Let $g := 1 - (f_1 + \dots + f_n)$. Then $g|U_k' = 0$ thus $g\omega$ can be continued to a_k by $g\omega(a_k) := 0$

$\rightarrow g\omega \in \mathcal{E}^{(1)}(X)$

~~By 10.20: $\iint_X d(g\omega) = 0$~~

By 10.20

$$\iint_X d(g\omega) = 0$$

Since ω is holomorphic $d\omega = 0$ on X'

$f_n \omega = \omega$ on $U_n' \cap X'$ $\Rightarrow$

$d(f_n\omega) = 0 \Rightarrow d(f_n\omega) \in \mathcal{E}^{(2)}(X)$ compactly supported in $U_n \setminus \{a_n\}$.

We have

$d(g\omega) = -\sum d(f_n\omega)$ therefore

$$0 = \iint_X d(g\omega) = -\sum \iint d(f_n\omega)$$

Claim: $\iint_X d(f_n\omega) = -2\pi i \operatorname{Res}_{a_n}(\omega)$:

Since $\operatorname{supp}(d(f_n\omega)) \subset U_n$ we only need ~~to~~ integrate over U_n

$\exists\ 0 < \varepsilon < R < 1$ s.t

$\operatorname{supp}(f_n) \subset \{|z_n| < R\}$ and $f_n|\{|z_n| \leq \varepsilon\} = 1 \Rightarrow$

$$\iint_X d(f_n\omega) = \iint_{\varepsilon \leq |z_n| \leq R} d(f_n\omega) = \int_{|z_n|=R} f_n\omega - \int_{|z_n|=\varepsilon} f_n\omega$$

$$= -\int_{|z_n|=\varepsilon} \omega = -2\pi i \operatorname{Res}_{a_n}(\omega)$$

By Residue theorem in $\mathbb{C}$

10.22 Corollary

Any nonconstant meromorphic fct. f on a comp. Riemann surface X has as many zeros as poles (counting multiplicities)