Residue Theorem

Before we start with cohomology I will show you two theorems as an addendum to last lecture.

Theorem X a Riemann-Surface and $\omega \in \mathcal{E}^{(1)}(X)$ is a differential form with compact support. Then

$$\iint_X d\omega = 0$$

Proof · We write $\omega = \omega_1 + \dots + \omega_n$ by multiplying ω with a partition of unity. Each ω_i has compact support which lies entirely in one chart.

- WLOG $X = \mathbb{C}$
- choose $R > 0$ st. $\operatorname{supp}(\omega) \subseteq B_R(0)$
- Then by Stokes

$$\iint_{\mathbb{C}} d\omega = \iint_{B_r(0)} d\omega = \int_{|z|=R} \omega = \int_{|z|=R} 0 = 0 \quad \square$$

Theorem (Residue) X compact RS. and $a_1, \dots, a_n$ are distinct points in X. Let $X' := X \setminus \{a_1, \dots, a_n\}$ then for every holomorphic 1-form $\omega \in \Omega(X')$ we have

$$\sum_{k=1}^{n} \operatorname{Res}_{a_k}(\omega) = 0$$

Recall: Let X RS., $Y \subseteq X$ open, $a \in Y$, ω holom. 1-form on $Y \setminus \{a\}$.
Let (U, z) be a coordinate nbhd of a st. $z(a) = 0$ then on $U \setminus \{a\}$ we write $\omega = f dz$ $f \in \mathcal{O}(U \setminus \{a\})$.

$f = \sum_{n=-\infty}^{\infty} c_n z^n$ Laurent expansion of f about a

$\Rightarrow \operatorname{Res}_a(\omega) := c_{-1}$

Proof: see p. 80

Cohomology

Def. Suppose X top. space, $\mathcal{F}$ is a sheaf of abelian groups on X. Given an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X st. $\bigcup_{i \in I} U_i = X$. For $q = 0, 1, 2, \dots$ define the q-the cochain group of $\mathcal{F}$ wrt. $\mathcal{U}$ as

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$$

Elements in $C^q(\mathcal{U}, \mathcal{F})$ are called q-cochains. Thus a q-chain is a family $\underbrace{(f_{i_0 \dots i_q})_{i_0, \dots, i_q \in I^{q+1}}}$ st. $f_{i_0 \dots i_q} \in \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$ $\forall (i_0, \dots, i_q) \in I^{q+1}$.

we just write $(f_{i_0 \dots i_q})$

Def. We define the boundary operator

$$\delta^0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$$

as follows:

i) For $(f_i) \in C^0(\mathcal{U}, \mathcal{F})$ let $\delta^0((f_i)) = (g_{ij})$ where $g_{ij} := f_j - f_i \in \mathcal{F}(U_i \cap U_j)$

ii) For $(f_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ let $\delta^1((f_{ij})) = (g_{ijk})$ where $g_{ijk} := \underbrace{f_{jk} - f_{ik} + f_{ij}}_{\in\, \mathcal{F}(U_i \cap U_j \cap U_k)}$

Then we can define:

$$Z^1(\mathcal{U}, \mathcal{F}) := \ker(\delta^1) \quad \text{and} \quad B^1(\mathcal{U}, \mathcal{F}) := \operatorname{Im}(\delta^0)$$

- We call elements of $Z^1(\mathcal{U}, \mathcal{F})$ 1-cocycles. By def.

  $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F}) \iff \delta^1((f_{ij})) = 0 \iff f_{ik} = f_{jk} + f_{ij}$ on $U_i \cap U_j \cap U_k$

  If $i = j = k \Rightarrow f_{ii} = 0$

  If $k = i \Rightarrow f_{ii} = f_{ji} + f_{ij} \Rightarrow f_{ij} = -f_{ji}$

- We call elements of $B^1(\mathcal{U}, \mathcal{F})$ 1-coboundaries / splitting cycle. In particular every coboundary is a cocycle. $(\operatorname{Im}(\delta^0) \subseteq \ker(\delta^1))$

  $(f_{ij}) \in B^1(\mathcal{U}, \mathcal{F}) \iff \exists (g_i)$ st. $f_{ij} = g_j - g_i$ on $U_i \cap U_j$ $\forall i, j \in I$

Def. $H^1(\mathcal{U},\mathcal{F}) := Z^1(\mathcal{U},\mathcal{F})/B^1(\mathcal{U},\mathcal{F})$ is called the **first cohomology group with coeff. in $\mathcal{F}$ wrt. $\mathcal{U}$.**

Note that two cocycles are cohomologues precisly if their difference is a coboundary.
Now we have the situation that the first cohomology group depends on the open covering. But we want a cohom. group independent of $\mathcal{U}$. For this we need some extra work.

- An covering $\mathcal{B} = (V_k)_{k\in K}$ is called **finer** then the covering $\mathcal{U} = (U_i)_{i\in I}$ denoted by $\mathcal{B} < \mathcal{U}$ if every V_k is contained in at least one U_i. Thus there is a mapping $\tau : K \to I$ st. $V_k \subset U_{\tau(k)}$ $\forall k \in K$.

Thus we define
$$t^{\mathcal{U}}_{\mathcal{B}} : Z^1(\mathcal{U},\mathcal{F}) \to Z^1(\mathcal{B},\mathcal{F})$$
$$(f_{ij}) \mapsto (g_{kl}) \quad \text{where } g_{kl} := f_{\tau(k)\tau(l)}|V_k \cap V_l$$

This map induces a homomorphism $H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{B},\mathcal{F})$ which we also denote by $t^{\mathcal{U}}_{\mathcal{B}}$

lemma The mapping $t^{\mathcal{U}}_{\mathcal{B}} : H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{U},\mathcal{F})$ is independent of the choice of the refing map $\tau : K \to I$ and is injective.

Proof

Independence:
- Suppose $\tilde{\tau} : K \to I$ another refining map. st. $V_k \subset U_{\tilde{\tau}(k)}$ $\forall k \in K$
- Suppose $(f_{ij}) \in Z^1(\mathcal{U},\mathcal{F})$ and let $g_{kl} := f_{\tau(k)\tau(l)}|V_k \cap V_l$ and $\tilde{g}_{kl} := f_{\tilde{\tau}(k)\tilde{\tau}(l)}|V_k \cap V_l$

Now we have to show that g_{kl} and $\tilde{g}_{kl}$ are cohomologues, which means their difference is a coboundary.

- Since $V_k \subset U_{\tau(k)} \cap U_{\tilde{\tau}(k)}$ can define $h_k := f_{\tau(k)\tilde{\tau}(k)}|V_k \in \mathcal{F}(V_k)$
- On $V_k \cap V_l$ we have
$$g_{kl} - \tilde{g}_{kl} = f_{\tau(k)\tau(l)} - f_{\tilde{\tau}(k)\tilde{\tau}(l)}$$
$$= f_{\tau(k)\tau(l)} - f_{\tilde{\tau}(k)\tilde{\tau}(l)} + f_{\tau(l)\tilde{\tau}(k)} - f_{\tau(l)\tilde{\tau}(k)}$$
$$= h_k - h_l \implies \text{it's a coboundary}$$

Injective: see p. 99 $\square$

Remark: For three open coverings $\mathcal{M} < \mathcal{B} < \mathcal{U}$ we have that $t^{\mathcal{B}}_{\mathcal{M}} \circ t^{\mathcal{U}}_{\mathcal{B}} = t^{\mathcal{U}}_{\mathcal{M}}$.

Def. $H^1(X, \mathcal{F}) := \bigcup_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) /\sim$

For $\xi \in H^1(\mathcal{U}, \mathcal{F})$ and $\eta \in H^1(\mathcal{U}', \mathcal{F})$ we have that

$\xi \sim \eta \iff \exists$ open covering $\mathcal{B} < \mathcal{U}$ and $\mathcal{B} < \mathcal{U}'$ s.t.

$$t^{\mathcal{U}}_{\mathcal{B}}(\xi) = t^{\mathcal{U}'}_{\mathcal{B}}(\eta)$$

If we take two elements x, y in $H^1(X, \mathcal{F})$ which are represented by $\xi \in H^1(\mathcal{U}, \mathcal{F})$ and $\eta \in H^1(\mathcal{U}', \mathcal{F})$. Then is $x+y$ represented by $t^{\mathcal{U}}_{\mathcal{B}}(\xi) + t^{\mathcal{U}'}_{\mathcal{B}}(\eta) \in H^1(\mathcal{B}, \mathcal{F})$ where $\mathcal{B}$ is a common refinment of $\mathcal{U}$ and $\mathcal{U}'$! One can easily check that this def. is independent of all the choices we made.

If $\mathcal{F}$ is a sheaf of VS. then $H^1(\mathcal{U}, \mathcal{F})$ and $H^1(X, \mathcal{F})$ are also VS.

$H^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is inj. by lemma. $\Rightarrow H^1(X, \mathcal{F}) = 0 \iff H^1(\mathcal{U}, \mathcal{F}) = 0 \quad \forall \mathcal{U}$.

Theorem X a R.S. and $\mathcal{E}$ is the sheaf of diff. functions on X.

Then $H^1(X, \mathcal{E}) = 0$.

Proof sketch | By §23 its valid to prove it under the assumtion that X has a countable Top. every RS has a countable top

- take arbitrary open covering $\mathcal{U} = (U_i)_{i \in I}$
- take partition of unity subordinate to $\mathcal{U}$ ie. $(\psi_i)_{i \in I} \in \mathcal{E}(X)$.
- We show that $H^1(\mathcal{U}, \mathcal{F}) = 0$ which means $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{E})$ splits.
- set $g_i := \sum_{j \in I} \psi_j f_{ij}$

$\hookrightarrow g_i - g_j = \sum_{k \in I} \psi_k f_{ik} - \sum_{k \in I} \psi_k f_{jk} = \sum_{k \in I} \psi_k (f_{ik} - f_{jk}) \overset{\text{properties}}{=} f_{ij}$ on $U_i \cap U_j$

$\Rightarrow (f_{ij})$ is a coboundary.

$\square$

Theorem X simply connected RS. then

i) $H^1(X, \mathbb{C}) = 0$ — sheaf of locally const. function with values in $\mathbb{C}$ or $\mathbb{Z}$. See p. 41 6.4e.

ii) $H^1(X, \mathbb{Z}) = 0$

Proof. i) • $\mathcal{U}$ open covering of X and $(c_{ij}) \in Z^1(\mathcal{U}, \mathbb{C})$.

• Since $Z^1(\mathcal{U}, \mathcal{F}) \subseteq Z^1(\mathcal{U}, \mathcal{E})$ and $H^1(\mathcal{U}, \mathcal{E}) = 0 \Rightarrow \exists (f_i) \in C^0(\mathcal{U}, \mathcal{E})$ coboundary st. $c_{ij} = f_i - f_j$ on $U_i \cap U_j$

• But $dc_{ij} = 0$ ← since loc. const. $\Rightarrow df_i = df_j$ on $U_i \cap U_j \Rightarrow \exists \omega \in \mathcal{E}^{(1)}(X)$ st. $\omega|U_i = df_i$. Since $ddf_i = 0 \Rightarrow \omega$ closed

$\Rightarrow \exists f \in \mathcal{E}(X)$ st. $df = \omega$ — here we use that X is simply connected

• Set $c_i := f_i - f|U_i$.

Since $dc_i = df_i - df = \omega - \omega = 0$ on U_i, c_i locally const.

ie. $(c_i) \in C^0(U, \mathbb{C})$.

• On $U_i \cap U_j$ one has $c_{ij} = f_i - f_j = (f_i - f) - (f_j - f) = c_i - c_j \Rightarrow c_{ij}$ splits

ii) use i) $\square$

Theorem (Leray) Suppose $\mathcal{F}$ is a sheaf of abelian groups on the top. space X and $\mathcal{U} = (U_i)_{i \in I}$ open covering of X st. $H^1(U_i, \mathcal{F}) = 0 \ \forall i \in I$. Then $H^1(X, \mathcal{F}) \cong H^1(\mathcal{U}, \mathcal{F})$

$\mathcal{U}$ is called Leray covering

~~45min~~

Proof • $t^{\mathcal{U}}_{\mathcal{B}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{B}, \mathcal{F})$

It is sufficient to show that for every open covering $\mathcal{B} = (V_\alpha)_{\alpha \in A}$ which is finer then $\mathcal{U} = (U_i)_{i \in I}$ this map is an isom. We already know it's injective.

• Suppose $\tau : A \to I$ is a refinig map. To prove surjectivity we show that for any given $(f_{\alpha\beta}) \in Z^1(\mathcal{B}, \mathcal{F})$ $\exists F_{ij} \in Z^1(\mathcal{U}, \mathcal{F})$ st. $F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta}$ is cohomolgues to 0 relative to $\mathcal{B}$.

• The family $\overbrace{(U_i \cap V_\alpha)_{\alpha \in A}}^{U_i \cap \mathcal{B}}$ is an open covering of U_i.

$H^1(U_i \cap \mathcal{B}, \mathcal{F}) = 0$ by assumtion $\Rightarrow \exists g_{i\alpha} \in \mathcal{F}(U_i \cap V_\alpha)$

st. $f_{\alpha\beta} = g_{i\alpha} - g_{i\beta}$ on $U_i \cap V_\alpha \cap V_\beta$

On $U_i \cap U_j \cap V_\alpha \cap V_\beta$ $\quad g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}$

$\Rightarrow \exists F_{ij} \in \mathcal{F}(U_i \cap U_j)$ st. $F_{ij} = g_{j\alpha} - g_{i\alpha}$ on $U_i \cap U_j \cap V_\alpha$

$\llcorner$ II Axiom of sheaf

$$\begin{array}{c} F_{ij} \\ \cap \\ Z^1(\mathcal{U}, \mathcal{F}) \end{array}$$

Def. $h_\alpha := g_{\tau(\alpha)\alpha} | V_\alpha \in \mathcal{F}(V_\alpha)$

$\Rightarrow$ On $V_\alpha \cap V_\beta$ $F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} = (g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}) - (g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta})$

$$= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} = h_\beta - h_\alpha \quad \Rightarrow \text{it splits}$$

$\square$

Def. Suppose $\mathcal{F}$ is a sheaf of abelian groups on a top. space X and $\mathcal{U} = (U_i)_{i \in I}$ an open covering of X.

Set $\quad Z^0(\mathcal{U}, \mathcal{F}) = \ker(\delta^0) \quad B^0(\mathcal{U}, \mathcal{F}) = 0$

$$H^0(\mathcal{U}, \mathcal{F}) := Z^0 / B^0 = Z^0$$

An element in $C^0(\mathcal{U}, \mathcal{F})$ (zero-th cohomology group wrt. $\mathcal{U}$) belongs to $Z^0(\mathcal{U}, \mathcal{F}) \Leftrightarrow f_i | U_i \cap U_j = f_j | U_i \cap U_j \quad \forall i, j \in I$

By the Axiom II of the sheaf f_i piece together to one global $f \in \mathcal{F}(X)$.

$$H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$$

$\hookrightarrow$ independent of $\mathcal{U}$ $\Rightarrow H^0(X, \mathcal{F}) := \mathcal{F}(X)$

Claim Let X be a compact RS. $\Rightarrow H^1(X, \mathbb{C})$ is a finite dim vs.

Proof Exercise.

Lemma (Dolbeault's) Suppose $g \in \mathcal{E}(\mathbb{C})$ has compact support. Then there exists a function $f \in \mathcal{E}(\mathbb{C})$ st.

$$\frac{\partial f}{\partial \bar{z}} = g$$

<u>Proof</u> $f : \mathbb{C} \to \mathbb{C}$ $\quad f(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z-\zeta} dz \wedge d\bar{z}$

We have a sing. when $z = \zeta$.
To show that the integral exists and depends differentiably on ζ.
For this we introduce polar coordinates.

- $z = \zeta + re^{i\theta}$
- $dz \wedge d\bar{z} = -2i dx \wedge dy = -2ir dr \wedge d\theta$

$\Rightarrow \quad f(\zeta) = -\frac{1}{\pi} \iint g(\zeta + re^{i\theta}) e^{-i\theta} dr d\theta$

- We integrate over $[0,R] \times [0,2\pi]$ for R suff. large.
- $$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = -\frac{1}{\pi} \iint \frac{\partial g(\zeta + re^{i\theta})}{\partial \bar{\zeta}} e^{-i\theta} dr d\theta$$

we need the ball because of the $\frac{1}{z}$.

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \iint_{B_\varepsilon} \frac{\partial g(\zeta + z)}{\partial \bar{\zeta}} \frac{1}{z} dz \wedge d\bar{z} \qquad \text{where } B_\varepsilon := \{ z \in \mathbb{C} \mid \varepsilon \leq |z| \leq R \}$$

Then we can simplify:

$$\frac{\partial g(\zeta + z)}{\partial \bar{\zeta}} \frac{1}{z} = \frac{\partial g(\zeta + z)}{\partial \bar{z}} \frac{1}{z} = \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta + z)}{z} \right)$$

- for $z \neq 0$ we have

$$\frac{\partial f(\zeta)}{\partial \bar{\zeta}} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \iint_{B_\varepsilon} \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta+z)}{z} \right) dz \wedge d\bar{z} = -\lim_{\varepsilon \to 0} \iint_{B_\varepsilon} d\omega = (*)$$

where $\omega = \frac{1}{2\pi i} \frac{g(\zeta + z)}{z} dz$

- $$\overset{\text{Stokes}}{\Rightarrow} \frac{\partial f}{\partial \bar{\zeta}} = -\lim_{\varepsilon \to 0} \iint_{B_\varepsilon} d\omega = \lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} \omega = (*)$$

$$\Rightarrow \frac{\partial f}{\partial \bar{\zeta}}(\zeta) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} g(\zeta + \varepsilon e^{i\theta}) d\theta = g(\zeta)$$

$\square$

15min for statement and proof.

Note that there is a theorem where we can drop the assumtion that g has compact support.

Corollary Suppose $X := \{z\in\mathbb{C} \mid |z|<R\}$, $0<R\le\infty$. Then given any $g\in\mathcal{E}(X)$ $\exists f\in\mathcal{E}(X)$ st. $\Delta f = g$, where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar z}$

Proof. Pick $\overset{\mathcal{E}(X)}{f_1}$ st. $\frac{\partial}{\partial\bar z} = g$ and $f_2\in\mathcal{E}(X)$ st. $\frac{\partial}{\partial\bar z}f_2 = \bar f_1 \Rightarrow \frac{\partial}{\partial z}\bar F_2 = f_1$

set $f = \frac{1}{4}\bar f_2 \quad \Rightarrow \Delta f = \frac{\partial^2\bar F_2}{\partial z\partial\bar z} = \frac{\partial}{\partial\bar z}\left(\frac{\partial\bar F_2}{\partial z}\right) = \frac{\partial}{\partial\bar z}f_1 = g$ □

Thm Suppose $X := \{z\in\mathbb{C}\mid |z|<R\}$, $0<R\le\infty$. Then $H^1(X,\mathcal{O})=0$.

Proof. • Suppose $\mathcal{U}=(U_i)_{i\in I}$ open covering of X and let $(f_{ij})\in Z^1(\mathcal{U},\mathcal{O})$

• Since $Z^1(\mathcal{U},\mathcal{O})\subset Z^1(\mathcal{U},\mathcal{E})$ and $H^1(X,\mathcal{E})=0$

$\Rightarrow \exists (g_i)\in C^0(\mathcal{U},\mathcal{E})$ st. $f_{ij} = g_i - g_j$ on $U_i\cap U_j$.

• Since $\underbrace{\frac{\partial}{\partial\bar z}f_{ij}=0}_{f_{ij}\text{ is holom.}} \Rightarrow \frac{\partial}{\partial\bar z}g_i = \frac{\partial}{\partial\bar z}g_j$ on $U_i\cap U_j$ $\Rightarrow \exists h\in\mathcal{E}(X)$ with

$h|U_i = \frac{\partial}{\partial\bar z}g_i$. By Dolbeault we find $g\in\mathcal{E}(X)$ st. $\frac{\partial}{\partial\bar z}g = h$

• Define $f_i := g_i - g$, $\frac{\partial}{\partial\bar z}f_i = \frac{\partial}{\partial\bar z}g_i - \frac{\partial}{\partial\bar z}g = 0 \Rightarrow f_i$ holom. $\Rightarrow (f_i)\in C^0(\mathcal{U},\mathcal{O})$

• $f_{ij} = g_i - g_j = f_i - f_j \Rightarrow (f_{ij})$ splits. □

Theorem $H^1(\mathbb{P}^1,\mathcal{O})=0$

Proof • We set $U_1 := \mathbb{P}^1\setminus\infty$ and $U_2 := \mathbb{P}^1\setminus\{0\}$

• Note that $U_1=\mathbb{C}$ and U_2 is biholomorphic to $\mathbb{C}$.

$\Rightarrow H^1(U_i,\mathcal{O})=0$ $i=1,2$ $\Rightarrow \mathcal{U}=(U_1,U_2)$ is a Leray covering of $\mathbb{P}^1$

$\overset{\text{Leray}}{\Rightarrow} H^1(\mathbb{P}^1,\mathcal{O}) = H^1(\mathcal{U},\mathcal{O})$ Thus it's sufficient to show that every $(f_{ij})\in Z^1(\mathcal{U},\mathcal{O})$ splits.

• It's enough to find $f_i\in\mathcal{O}(U_i)$ $i=1,2$ st. $f_{12}=f_1-f_2$ on $U_1\cap U_2=\mathbb{C}^*$

Since $f_{11}=0$ by properties

• Let $f_{12}(z) = \sum_{n=-\infty}^{\infty}c_nz^n$ the Laurent expansion of f_{12} on $\mathbb{C}^*$

Set $f_1(z) = \sum_{n=0}^{\infty}c_nz^n$ and $f_2(z) := -\sum_{n=-\infty}^{-1}c_nz^n \Rightarrow f_i\in\mathcal{O}(U_i)$ and $f_{12}=f_1-f_2$

 □