A Finiteness Theorem

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The main reference of this work is the book "Lectures on Riemann Surfaces" by Otto Forster.

1 Preliminaries

In this section we prove that for any compact Riemann surface X the cohomology group $H^1(X, \mathcal{O})$ is a finite dimensional complex vector space. Its dimension is called the genus of X. One of the consequences of the finiteness theorem is the existence of non-constant meromorphic functions on every compact Riemann surface.

We start by recalling what is the first cohomology group $H^1(X, \mathcal{O})$.

Definition 1.1. Suppose X is a topological space and $\mathscr{F}$ is a sheaf of abelian groups on X. Also suppose that an open covering of X is given i.e. a family $\mathcal{U} = (U_i)_{i\in I}$ of open subsets of X such that $\bigcup_{i\in I} U_i = X$. For $q = 0, 1, 2, \dots$ define the **q-th cochain group of $\mathscr{F}$**, with respect to $\mathcal{U}$, as

$$C^q(\mathcal{U}, \mathscr{F}) \coloneqq \bigtimes_{(i_0,\dots,i_q)\in I^{q+1}} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_q}).$$

The elements of $C^q(\mathcal{U}, \mathscr{F})$ are called **q-cochains**. Thus a q-cochain is a family

$$(f_{i_0\dots i_q})_{i_0,\dots\dots,i_q\in I^{q+1}} \text{ such that } f_{i_0\dots i_q} \in \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_q})$$

for all $(i_0, \dots, i_q) \in I^{q+1}$. The addition of two cochains is defined component-wise.

Definition 1.2. Define the **coboundary operators** as the group homomorphisms

$$\begin{aligned}
\delta : C^0(\mathcal{U}, \mathscr{F}) &\to C^1(\mathcal{U}, \mathscr{F}),\\
(f_i)_{i\in I} &\mapsto (f_j - f_i)_{i,j\in I} \subset \mathscr{F}(U_i \cap U_j)\\
\delta : C^1(\mathcal{U}, \mathscr{F}) &\to C^2(\mathcal{U}, \mathscr{F}),\\
(f_{ij})_{i,j\in I} &\mapsto (f_{jk} - f_{ik} + f_{ij})_{i,j,k\in I} \subset \mathscr{F}(U_i \cap U_j \cap U_k)
\end{aligned}$$

and define

$$\begin{aligned}
Z^1(\mathcal{U}, \mathscr{F}) &\coloneqq \mathrm{Ker}(C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathscr{F})),\\
B^1(\mathcal{U}, \mathscr{F}) &\coloneqq \mathrm{Im}(C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F})).
\end{aligned}$$

The elements of $Z^1(\mathcal{U}, \mathscr{F})$ are called **1-cocycles** and the elements of $B^1(\mathcal{U}, \mathscr{F})$ are called **1-coboundaries**. In particular every coboundary is a cocycle.

Definition 1.3. The quotient group

$$H^1(\mathcal{U}, \mathscr{F}) \coloneqq Z^1(\mathcal{U}, \mathscr{F})/B^1(\mathcal{U}, \mathscr{F})$$

is called the **1st cohomology group** with coefficients in $\mathscr{F}$ with respect to the covering $\mathcal{U}$. Its elements are called **cohomology classes** and two cocycles which belong to the same cohomology class are called cohomologous.

Theorem 1.1. Suppose X is a Riemann surface and $\mathscr{E}$ is the sheaf of differentiable functions on X. Then $H^1(X, \mathscr{E}) = 0$.

Theorem 1.2 (Leray)**.** Suppose $\mathscr{F}$ is a sheaf of abelian groups on the topological space X and $\mathcal{U} = (U_i)_{i \in I}$ is an open covering of X such that $H^1(U_i, \mathscr{F}) = 0$ for every $i \in I$. Then

$$H^1(X, \mathscr{F}) \cong H^1(\mathcal{U}, \mathscr{F}).$$

Theorem 1.3. Suppose $X \coloneqq \{z \in \mathbb{C} : |z| < R\},\ 0 < R \leq \infty$ and $g \in \mathscr{E}(X)$. Then there exists $f \in \mathscr{E}(X)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Theorem 1.4. Suppose $X \coloneqq \{z \in \mathbb{C} : |z| < R\},\ 0 < R \leq \infty$. Then $H^1(X, \mathcal{O}) = 0$.

Theorem 1.5. For the Riemann sphere $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Finally, recall a theorem we will need later.

Definition 1.4. A topological vector space E is called a **Fréchet space** if the following hold:

(i) The topology of E is Hausdorff and can be defined by a countable family of semi-norms;

(ii) E is complete i.e. every Cauchy sequence in E is convergent.

Theorem 1.6 (Banach)**.** Suppose E and F are Fréchet spaces and $f : E \to F$ is a continuous linear surjective mapping. Then f is open.

Corollary. Suppose E and F are Banach spaces and $f : E \to F$ is a continuous linear surjective mapping. Then there exists a constant $C > 0$ such that for every $y \in F$ there is an $x \in E$ with

$$f(x) = y \quad \text{and} \quad \|x\| \leq C\|y\|.$$

2 The L^2-Norm for Holomorphic Functions

Suppose $D \subset \mathbb{C}$ is an open set. Given a holomorphic function $f \in \mathcal{O}(D)$ define its L^2-norm by

$$\|f\|_{L^2(D)} \coloneqq \left(\iint_D |f(x+iy)|^2 \, \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

Then $\|f\|_{L^2(D)} \in \mathbb{R}_+ \cup \{\infty\}$. If $\|f\|_{L^2(D)} < \infty$, then f is called square integrable. We denote by $L^2(D, \mathcal{O})$ the vector space of all square integrable holomorphic functions on D. If

$$\mathrm{Vol}(D) \coloneqq \iint_D \mathrm{d}x\mathrm{d}y < \infty$$

then for every bounded function $f \in \mathcal{O}(D)$ one has

$$\|f\|_{L^2(D)} \leq \sqrt{\mathrm{Vol}(D)}\|f\|_D$$

where $\|f\|_D \coloneqq \sup\{|f(z)| : z \in D\}$ denotes the supremum norm.

For $f, g \in L^2(D, \mathcal{O})$ one can define an inner product $\langle f, g \rangle \in \mathbb{C}$ by

$$\langle f, g \rangle \coloneqq \iint_D f\bar{g} \, \mathrm{d}x\mathrm{d}y.$$

The integral exists because for every $z \in D$:

$$\left| f(z)\overline{g(z)} \right| \leq \frac{1}{2}\left(|f(z)|^2 + |g(z)|^2\right).$$

With this inner product $L^2(D, \mathcal{O})$ is a unitary vector space and in particular has a well-defined notion of orthogonality. Now suppose $B \coloneqq B(a, r) \coloneqq \{z \in \mathbb{C} : |z - a| < r\}$ is the disk with center a and radius $r > 0$. Then the monomials $(\psi_n)_{n \in N}$ given by

$$\psi_n(z) \coloneqq (z - a)^n$$

form an orthogonal system in $L^2(B, \mathcal{O})$ and one can easily check, using polar coordinates, that

$$\|\psi_n\|_{L^2(B)} = \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}} \text{ for every } n \in \mathbb{N}.$$

If $f \in L^2(B, \mathcal{O})$ and

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

is the Taylor series of f about a, it follows from Pythagoras that

$$\|f\|^2_{L^2(B)} = \sum_{n=0}^{\infty} \frac{\pi r^{2n+2}}{n+1} |c_n|^2. \tag{$*$}$$

Theorem 2.1. Suppose $D \subset \mathbb{C}$ is open, $r > 0$ and

$$D_r \coloneqq \{z \in \mathbb{C} : B(z, r) \subset D\}$$

is the set of points in D whose distance from the boundary is greater than or equal to r. Then for every $f \in L^2(D, \mathcal{O})$ one has

$$\|f\|_{D_r} \leq \frac{1}{\sqrt{\pi} r}\|f\|_{L^2(D)}.$$

Proof. Suppose $a \in D_r$ and $f(z) = \sum c_n (z - a)^n$ is the Taylor series of f about a. Using $(*)$ one gets

$$|f(a)| = |c_0| \leq \frac{1}{\sqrt{\pi} r}\|f\|_{L^2(B(a,r))} \leq \frac{1}{\sqrt{\pi} r}\|f\|_{L^2(D)}.$$

Since $\|f\|_{D_r} = \sup\{|f(a)| : a \in D_r\}$, the result follows. □

In particular, it follows from Theorem (2.1) that if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D,\mathcal{O})$, then the sequence converges uniformly on every compact subset of D. Thus the limit function is holomorphic. Hence $L^2(D,\mathcal{O})$ is complete and thus is a Hilbert space.

The following lemma may be viewed as a certain generalization of Schwarz' Lemma.

Lemma 2.1. Suppose $D' \Subset D$ are open subsets of $\mathbb{C}$. Then given any $\varepsilon > 0$, there exists a closed vector subspace $A \subset L^2(D,\mathcal{O})$ of finite codimension such that for every $f \in A$:

$$\|f\|_{L^2(D')} \leq \varepsilon \|f\|_{L^2(D)}.$$

Proof. Since $\overline{D'}$ is compact and lies in D, there exist $r > 0$ and finitely many points $a_1, \ldots, a_k \in D$ with the following properties:

(i) $B(a_j, r) \subset D$ for $j = 1, \ldots, k$;

(ii) $D' \subset \bigcup_{j=1}^{k} B(a_j, r/2)$.

Choose n so large that $2^{-n-1}k \leq \varepsilon$. Let A be the set of all functions $f \in L^2(D,\mathcal{O})$ which vanish at every point a_j at least to order n. Then A is a closed vector subspace of $L^2(D,\mathcal{O})$ of codimension $\leq kn$. Let $f \in A$. Then f has a Taylor series about a_j

$$f(z) = \sum_{\nu=n}^{\infty} c_\nu (z-a_j)^\nu.$$

For every $0 < \rho \leq r$ one has

$$\|f\|^2_{L^2(B(a_j,\rho))} = \sum_{\nu=n}^{\infty} \frac{\pi\rho^{2n+2}}{\nu+1} |c_\nu|^2,$$

from which it follows that

$$\|f\|_{L^2(B(a_j,r/2))} \leq 2^{-n-1} \|f\|_{L^2(B(a_j,r))}.$$

Using (i) and (ii) one has

$$\|f\|_{L^2(B(a_j,r))} \leq \|f\|_{L^2(D)}$$

and

$$\|f\|_{L^2(D')} \leq \sum_{j=1}^{k} \|f\|_{L^2(B(a_j,r/2))}.$$

Thus

$$\|f\|_{L^2(D')} \leq k \cdot 2^{-n-1} \|f\|_{L^2(D)} \leq \varepsilon \|f\|_{L^2(D)}.$$

□

3 Square Integrable Cochains

Suppose X is a Riemann surface. Choose a finite family (U_i^*, z_i), $i = 1, \dots, n$, of charts on X such that every $z_i(U_i^*) \subset \mathbb{C}$ is a disk (note however that we are not assuming that $\mathcal{U}^* := (U_i^*)_{1\le i\le n}$ is a covering of X). Suppose $U_i \subset U_i^*$ are open subsets and set $\mathcal{U} := (U_i)_{1\le i\le n}$. We introduce L^2-norms on the cochain groups $C^0(\mathcal{U}, \mathcal{O})$ and $C^1(\mathcal{U}, \mathcal{O})$, defined on the space

$$|\mathcal{U}| := U_1 \cup \cdots \cup U_n,$$

in the following way:

(i) For $\eta = (f_i)_{i=1}^n \in C^0(\mathcal{U}, \mathcal{O})$ let

$$\|\eta\|^2_{L^2(\mathcal{U})} := \sum_{i=1}^n \|f_i\|^2_{L^2(U_i)};$$

(ii) For $\xi = (f_{ij})_{i,j=1}^n \in C^1(\mathcal{U}, \mathcal{O})$ let

$$\|\xi\|^2_{L^2(\mathcal{U})} := \sum_{i,j=1}^n \|f_{ij}\|^2_{L^2(U_i\cap U_j)}.$$

Here the norms of f_i and f_{ij} are calculated with respect to the chart (U_i^*, z_i) i.e.

$$\|f_i\|_{L^2(U_i)} := \left\|f_i \circ z_i^{-1}\right\|_{L^2(z_i(U_i))},$$
$$\|f_{ij}\|_{L^2(U_i\cap U_j)} := \left\|f_{ij} \circ z_i^{-1}\right\|_{L^2(z_i(U_i\cap U_j))}.$$

The set of q-cochains having finite norm is a vector subspace $C^q_{L^2}(\mathcal{U}, \mathcal{O}) \subset C^q(\mathcal{U}, \mathcal{O})$, $q = 0, 1$, and these subspaces are Hilbert spaces. The cocycles in $C^1_{L^2}(\mathcal{U}, \mathcal{O})$ form a closed vector subspace which we denote by $Z^1_{L^2}(\mathcal{U}, \mathcal{O})$.

Remark (1)**.** If $V_i \Subset U_i$, $i = 1, \dots, n$, are relatively compact open subsets which compose the family $\mathcal{V} := (V_i)_{1\le i\le n}$, then, to simplify the notation, we will write $\mathcal{V} \ll \mathcal{U}$. For any cochain $\xi \in C^q(\mathcal{U}, \mathcal{O})$ one has $\|\xi\|_{L^2(\mathcal{V})} < \infty$. It then follows directly from Lemma (2.1) that given any $\varepsilon > 0$, there exists a closed vector subspace $A \subset Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ of finite codimension such that for every $\xi \in A$:

$$\|\xi\|_{L^2(\mathcal{V})} \le \varepsilon \|\xi\|_{L^2(\mathcal{U})}.$$

Lemma 3.1. Suppose X is a Riemann surface and $\mathcal{U}^*$ is a finite family of charts on X as before. Further suppose that one has $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$ i.e. fixed shrinkings of $\mathcal{U}^*$ are given. Then there exists a constant $C > 0$ such that for every cocycle $\xi \in Z^1_{L^2}(\mathcal{V}, \mathcal{O})$ there exist elements $\zeta \in Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ and $\eta \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$ with

$$\zeta = \xi + \delta(\eta) \quad \text{on } \mathcal{W} \tag{i}$$

and

$$\max\left\{\|\zeta\|_{L^2(\mathcal{U})}, \|\eta\|_{L^2(\mathcal{W})}\right\} \le C\|\xi\|_{L^2(\mathcal{V})}. \tag{ii}$$

Proof. (i) Suppose $\xi = (f_{ij}) \in Z^1_{L^2}(\mathcal{V}, \mathcal{O})$ is given. Forgetting for the moment the restriction on the norms, we first construct $\zeta \in Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ and $\eta \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$ such that $\zeta = \xi + \delta(\eta)$ on $\mathcal{W}$. By Theorem (1.1) there exists a cochain $(g_i) \in C^0(\mathcal{V}, \mathscr{E})$ such that

$$f_{ij} = g_j - g_i \quad \text{on } V_i \cap V_j.$$

Since $\frac{\partial f_{ij}}{\partial \bar{z}} \mathrm{d}z = 0$, one has $\frac{\partial g_i}{\partial \bar{z}} \mathrm{d}z = \frac{\partial g_j}{\partial \bar{z}} \mathrm{d}z$ on $V_i \cap V_j$, and thus there exists a differential form $\omega \in \mathscr{E}^{0,1}(|\mathcal{V}|)$ with $\omega | V_i = \frac{\partial g_i}{\partial \bar{z}} \mathrm{d}z$. Since $|\mathcal{W}| \Subset |\mathcal{V}|$, there exists a function $\psi \in \mathscr{E}(X)$ with

$$\mathrm{Supp}(\psi) \subset |\mathcal{V}| \quad \text{and } \psi|_{|\mathcal{W}|} = 1.$$

Hence $\psi\omega$ can be considered as an element of $\mathscr{E}(|\mathcal{U}^*|)$. By Theorem (1.3) there exist functions $h_i \in \mathscr{E}(U_i^*)$ such that

$$\frac{\partial h_i}{\partial \bar{z}} \mathrm{d}z = \psi\omega \quad \text{on } U_i^*.$$

Because $\frac{\partial h_i}{\partial \bar{z}} \mathrm{d}z = \frac{\partial h_j}{\partial \bar{z}} \mathrm{d}z$ on $U_i^* \cap U_j^*$, it follows that

$$F_{ij} := h_j - h_i \in \mathcal{O}(U_i^* \cap U_j^*).$$

Set $\zeta := (F_{ij})|_{\mathcal{U}}$. Since $\mathcal{U} \ll \mathcal{U}^*$, one has $\zeta \in Z^1_{L^2}(\mathcal{U}, \mathcal{O})$. On W_i one has

$$\frac{\partial h_i}{\partial \bar{z}} \mathrm{d}z = \psi\omega = \omega = \frac{\partial g_i}{\partial \bar{z}} \mathrm{d}z,$$

thus $h_i - g_i$ is holomorphic on W_i. Since $h_i - g_i$ is also bounded on W_i, one has

$$\eta := (h_i - g_i)|_{\mathcal{W}} \in C^0_{L^2}(\mathcal{W}, \mathcal{O}).$$

Now $F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i)$ on $W_i \cap W_j$ and thus

$$\zeta - \xi = \delta(\eta) \quad \text{on } \mathcal{W}.$$

(ii) In order to get the desired estimate on the norms, we consider the Hilbert space

$$H := Z^1_{L^2}(\mathcal{U}, \mathcal{O}) \times Z^1_{L^2}(\mathcal{V}, \mathcal{O}) \times C^0_{L^2}(\mathcal{W}, \mathcal{O})$$

with the norm

$$\|(\zeta, \xi, \eta)\|_H := \left(\|\zeta\|^2_{L^2(\mathcal{U})} + \|\xi\|^2_{L^2(\mathcal{V})} + \|\eta\|^2_{L^2(\mathcal{W})} \right)^{1/2}.$$

Let $L \subset H$ be the subspace

$$L := \{(\zeta, \xi, \eta) \in H : \zeta = \xi + \delta(\eta) \text{ on } \mathcal{W}\}.$$

Since L is closed in H, it is also a Hilbert space. From (i) the continuous linear mapping

$$\pi : L \to Z^1_{L^2}(\mathcal{V}, \mathcal{O}),$$
$$(\zeta, \xi, \eta) \mapsto \xi$$

is surjective. By the Theorem of Banach (1.6) the mapping π is open. Thus there exists a constant $C > 0$ such that for every $\xi \in Z^1_{L^2}(\mathcal{V}, \mathcal{O})$ there exists $x = (\zeta, \xi, \eta) \in L$ with $\pi(x) = \xi$ and $\|x\|_H \le C\|\xi\|_{L^2(\mathcal{V})}$. This constant then satisfies the desired conditions. □

4 A Finiteness Theorem

Lemma 4.1. Under the same assumptions as in Lemma (3.1), there exists a finite dimensional vector subspace $S \subset Z^1(\mathcal{U}, \mathcal{O})$ such that for every $\xi \in Z^1(\mathcal{U}, \mathcal{O})$ there exist elements $\sigma \in S$ and $\eta \in C^0(\mathcal{W}, \mathcal{O})$ such that

$$\sigma = \xi + \delta(\eta) \quad \text{on } \mathcal{W}.$$

Remark. The lemma says that the natural restriction mapping

$$H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{W}, \mathcal{O})$$

has a finite dimensional image. To see this, rewrite the mapping by the definition of the 1st cohomology group i.e.

$$Z^1(\mathcal{U}, \mathcal{O})/B^1(\mathcal{U}, \mathcal{O}) \to Z^1(\mathcal{W}, \mathcal{O})/B^1(\mathcal{W}, \mathcal{O})$$

and note that $\delta(\eta) \in B^1(\mathcal{W}, \mathcal{O})$ by definition of $B^1(\mathcal{W}, \mathcal{O})$ and, since the relation above holds on $\mathcal{W}$, we have that the restriction has finite dimensional image since S is finite dimensional.

Proof. Suppose C is the constant in Lemma (3.1) and set $\varepsilon := 1/2C$. By Remark (1) there exists a finite codimensional closed vector subspace $A \subset Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ such that for every $\xi \in A$:

$$\|\xi\|_{L^2(\mathcal{V})} \le \varepsilon \|\xi\|_{L^2(\mathcal{U})}.$$

Let S be the orthogonal complement of A in $Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ i.e. $A \oplus S = Z^1_{L^2}(\mathcal{U}, \mathcal{O})$. Now suppose $\xi \in Z^1(\mathcal{U}, \mathcal{O})$ is arbitrary. Because $\mathcal{V} \ll \mathcal{U}$,

$$M := \|\xi\|_{L^2(\mathcal{V})} < \infty.$$

By Lemma (3.1) there exist $\zeta_0 \in Z^1_{L_2}(\mathcal{U}, \mathcal{O})$ and $\eta_0 \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$ such that

$$\zeta_0 = \xi + \delta(\eta_0) \quad \text{on } \mathcal{W}$$

and $\|\zeta_0\|_{L^2(\mathcal{U})} \le CM$, $\|\eta_0\|_{L^2(\mathcal{W})} \le CM$. Suppose that for $\xi_0 \in A$, $\sigma_0 \in S$,

$$\zeta_0 = \xi_0 + \sigma_0$$

is the orthogonal decomposition of ζ_0. We now construct, by induction, elements

$$\zeta_\nu \in Z^1_{L_2}(\mathcal{U}, \mathcal{O}), \quad \eta_\nu \in C^0_{L^2}(\mathcal{W}, \mathcal{O}), \quad \xi_\nu \in A, \quad \sigma_\nu \in S$$

with the following properties:

(i) $\zeta_\nu = \xi_{\nu-1} + \delta(\eta_\nu)$ on $\mathcal{W}$;

(ii) $\zeta_\nu = \xi_\nu + \sigma_\nu$;

(iii) $\|\zeta_\nu\|_{L^2(\mathcal{U})} \leq 2^{-\nu} CM, \quad \|\eta_\nu\|_{L^2(\mathcal{W})} \leq 2^{-\nu} CM.$

Consider the induction step from ν to $\nu+1$. Since $\zeta_\nu = \xi_\nu + \sigma_\nu$ is an orthogonal decomposition, one has

$$\|\xi_\nu\|_{L^2(\mathcal{U})} \leq \|\zeta_\nu\|_{L^2(\mathcal{U})} \leq 2^{-\nu} CM.$$

Thus, by Remark (1),

$$\|\xi_\nu\|_{L^2(\mathcal{V})} \leq \varepsilon \|\xi_\nu\|_{L^2(\mathcal{U})} \leq 2^{-\nu} \varepsilon CM \leq 2^{-\nu-1} M.$$

By Lemma (3.1) there exist elements $\zeta_{\nu+1} \in Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ and $\eta_{\nu+1} \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$ such that

$$\zeta_{\nu+1} = \xi_\nu + \delta(\eta_{\nu+1}) \quad \text{ on } \mathcal{W}$$

and

$$\max\left\{ \|\zeta_{\nu+1}\|_{L^2(\mathcal{U})}, \|\eta_{\nu+1}\|_{L^2(\mathcal{W})} \right\} \leq 2^{-\nu-1} CM.$$

Now one has an orthogonal decomposition $\zeta_{\nu+1} = \xi_{\nu+1} + \sigma_{\nu+1}$, where $\xi_{\nu+1} \in A$ and $\sigma_{\nu+1} \in S$, and thus the induction step is complete.

From the equation $\zeta_0 = \xi + \delta(\eta_0)$, together with equations (i) and (ii) up to $\nu = k$, one gets

$$\xi_k + \sum_{\nu=0}^{k} \sigma_\nu = \xi + \delta\left(\sum_{\nu=0}^{k} \eta_\nu \right) \quad \text{ on } \mathcal{W}. \tag{$**$}$$

From (ii) and (iii) it follows that

$$\max\left\{ \|\xi_\nu\|_{L^2(\mathcal{U})}, \|\sigma_\nu\|_{L^2(\mathcal{U})}, \|\eta_\nu\|_{L^2(\mathcal{W})} \right\} \leq 2^{-\nu} CM.$$

Hence $\lim_{k\to\infty} \xi_k = 0$ and the series

$$\sigma := \sum_{\nu=0}^{\infty} \sigma_\nu \in S$$

$$\eta := \sum_{\nu=0}^{\infty} \eta_\nu \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$$

converge. Finally from $(**)$ one gets $\sigma = \xi + \delta(\eta)$ on $\mathcal{W}$. □

Suppose X is a topological space, $Y \subset X$ is open and $\mathscr{F}$ is a sheaf of abelian groups on X. For every open covering $\mathcal{U} = (U_i)_{i\in I}$ of X, $\mathcal{U} \cap Y := (U_i \cap Y)_{i \in I}$ is an open covering of Y and the natural restriction mapping $Z^1(\mathcal{U}, \mathscr{F}) \to Z^1(\mathcal{U} \cap Y, \mathscr{F})$ induces a homomorphism

$$H^1(\mathcal{U}, \mathscr{F}) \to H^1(\mathcal{U} \cap Y, \mathscr{F}).$$

These homomorphisms for all $\mathcal{U}$ give rise to a restriction homomorphism

$$H^1(X, \mathscr{F}) \to H^1(Y, \mathscr{F}).$$

Theorem 4.1. Suppose X is a Riemann surface and $Y_1 \Subset Y_2 \subset X$ are open subsets. Then the restriction homomorphism

$$H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$$

has a finite dimensional image.

Proof. There exists a finite family of charts $(U_i^*, z_i)_{i=1}^n$ on X and relatively compact open subsets $W_i \Subset V_i \Subset U_i \Subset U_i^*$ with the following properties:

(i) $Y_1 \subset \bigcup_{i=1}^n W_i =: Y' \Subset Y'' := \bigcup_{i=1}^n U_i \subset Y_2$;

(ii) all $z_i(U_i^*)$, $z_i(U_i)$ and $z_i(W_i)$ are disks in $\mathbb{C}$.

Let $\mathcal{U} := (U_i)_{i=1}^n$, $\mathcal{W} := (W_i)_{i=1}^n$. By Lemma (4.1) the restriction mapping

$$H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{W}, \mathcal{O})$$

has a finite dimensional image. By Theorem (1.4) we have

$$H^1(U_i, \mathcal{O}) = H^1(W_i, \mathcal{O}) = 0.$$

Thus, by Leray's Theorem (1.2),

$$H^1(\mathcal{U}, \mathcal{O}) = H^1\left(Y'', \mathcal{O}\right) \quad \text{and} \quad H^1(\mathcal{W}, \mathcal{O}) = H^1(Y', \mathcal{O}).$$

The restriction mapping $H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$ can be factored as follows:

$$H^1(Y_2, \mathcal{O}) \to H^1(Y'', \mathcal{O}) \to H^1(Y', \mathcal{O}) \to H^1(Y_1, \mathcal{O})$$

and hence the proof of the theorem is complete. □

5 Applications

Corollary. Suppose X is a compact Riemann surface. Then

$$\dim H^1(X, \mathcal{O}) < \infty.$$

Proof. Since X is compact, one can choose $Y_1 = Y_2 = X$ in the previous theorem. □

Definition 5.1. Suppose X is a compact Riemann surface. Then

$$g := \dim H^1(X, \mathcal{O})$$

is called the **genus** of X.

By Theorem (1.5) the Riemann sphere $\mathbb{P}^1$ has genus zero and by the previous Corollary every compact Riemann surface has finite genus.

Theorem 5.1. Suppose X is a Riemann surface and $Y \Subset X$ is a relatively compact open subset. Then for every point $a \in Y$ there exists a meromorphic function $f \in \mathscr{M}(Y)$ which has a pole at a and is holomorphic on $Y \setminus \{a\}$.

Proof. By Theorem (4.1)

$$k := \dim \operatorname{Im}(H^1(X, \mathcal{O}) \to H^1(Y, \mathcal{O})) < \infty.$$

Suppose (U_1, z) is a coordinate neighborhood of a with $z(a) = 0$. Set $U_2 := X \setminus \{a\}$. Then $\mathcal{U} = (U_1, U_2)$ is an open covering of X. The functions z^{-j} are holomorphic on $U_1 \cap U_2 = U_1 \setminus \{a\}$ and represent cocycles

$$\zeta_j \in Z^1(\mathcal{U}, \mathcal{O}), \quad j = 1, \ldots, k+1.$$

Since $\dim \operatorname{Im}(H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{U} \cap Y, \mathcal{O})) < k+1$, the cocycles

$$\zeta_j|_Y \ \in Z^1(\mathcal{U} \cap Y, \mathcal{C}), \quad j = 1, \ldots, k+1,$$

are linearly dependent modulo the coboundaries. Thus there exist complex numbers $c_1, \ldots, c_{k+1}$, not all zero, and a cochain $\eta = (f_1, f_2) \in C^0(\mathcal{U} \cap Y, \mathcal{O})$ such that

$$c_1\zeta_1 + \cdots + c_{k+1}\zeta_{k+1} = \delta(\eta) \quad \text{ with respect to } \mathcal{U} \cap Y$$

i.e.

$$\sum_{j=1}^{k+1} c_j z^{-j} = f_2 - f_1 \quad \text{ on } U_1 \cap U_2 \cap Y.$$

Hence there is a function $f \in \mathscr{M}(Y)$, which coincides with

$$f_1 + \sum_{j=1}^{k+1} c_j z^{-j}$$

on $U_1 \cap Y$ and which is equal to f_2 on $U_2 \cap Y = Y \setminus \{a\}$. This is the desired function. □

Corollary. Suppose X is a compact Riemann surface and $a_1, \ldots, a_n$ are distinct points on X. Then for any given complex numbers $c_1, \ldots, c_n \in \mathbb{C}$, there exists a meromorphic function $f \in \mathscr{M}(X)$ such that $f(a_i) = c_i$ for $i = 1, \ldots, n$.

Proof. For every pair $i \neq j$, by applying Theorem (5.1) in the case $Y = X$, one gets a function $f_{ij} \in \mathscr{M}(X)$ which has a pole at a_i but is holomorphic at a_j. Choose a constant $\lambda_{ij} \in \mathbb{C}^*$ such that $f_{ij}(a_k) \neq f_{ij}(a_j) - \lambda_{ij}$ for every $k = 1, \ldots, n$. Then the function

$$g_{ij} := \frac{f_{ij} - f_{ij}\,(a_j)}{f_{ij} - f_{ij}\,(a_j) + \lambda_{ij}} \in \mathscr{M}(X)$$

is holomorphic at the points a_k, $1 \leq k \leq n$, and satisfies $g_{ij}(a_i) = 1$ and $g_{ij}(a_j) = 0$. Now the functions

$$h_i := \prod_{\substack{j=1 \\ j \neq i}}^{n} g_{ij}, \quad i = 1, \ldots, n,$$

satisfy $h_i(a_j) = \delta_{ij}$ and thus

$$f := \sum_{i=1}^{n} c_i h_i$$

solves the problem. $\square$

We now note a few consequences of the finiteness theorem for noncompact Riemann surfaces.

Corollary. Suppose Y is a relatively compact open subset of a noncompact Riemann surface X. Then there exists a holomorphic function $f : Y \to \mathbb{C}$ which is not constant on any connected component of Y.

Proof. Choose a domain Y_1 such that $Y \Subset Y_1 \Subset X$ and a point $a \in Y_1 \setminus Y$. (Since X is non-compact and connected, $Y_1 \setminus Y$ is not empty.) Now apply Theorem (5.1) to Y_1 and the point a. $\square$

Theorem 5.2. Suppose X is a non-compact Riemann surface and $Y \Subset Y' \subset X$ are open subsets. Then

$$\operatorname{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0.$$

Proof. By Theorem (4.1) we already know that

$$L := \operatorname{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O}))$$

is a finite dimensional vector space. Choose cohomology classes $\xi_1, \ldots, \xi_n \in H^1(Y', \mathcal{O})$ such that their restrictions to Y span the vector space L. According to the previous corollary we may choose a function $f \in \mathcal{O}(Y')$ which is not constant on any connected component of Y'. Since $H^1(Y', \mathcal{O})$ is in a natural way a module over $\mathcal{O}(Y')$, the products $f\xi_\nu \in H^1(Y', \mathcal{O})$ are defined for every $\nu = 1, \ldots, n$. By the choice of the ξ_ν there exist constants $c_{\nu\mu} \in \mathbb{C}$ such that

$$f\xi_\nu = \sum_{\mu=1}^{n} c_{\nu\mu}\xi_\mu \quad \text{on } Y \text{ for } \nu = 1, \ldots, n. \tag{$\star$}$$

Set

$$F := \det\left(f\delta_{\nu\mu} - c_{\nu\mu}\right)_{1\le\nu,\ \mu\le n}.$$

Then F is a holomorphic function on Y' which is not identically zero on any connected component of Y'. From $(\star)$ it follows that

$$F\xi_\nu|_Y = 0 \quad \text{for } \nu = 1, \ldots, n \tag{$\star\star$}$$

An arbitrary cohomology class $\zeta \in H^1(Y', \mathcal{O})$ can be represented by a cocycle $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O})$, where $\mathcal{U} = (U_i)_{i\in I}$ is an open covering of Y' such that each zero of F is contained in at most one U_i. Thus for $i \neq j$ one has $F|_{U_i\cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$. Hence there exists a cocycle $(g_{ij}) \in Z^1(\mathcal{U}, \mathcal{O})$ such that $f_{ij} = Fg_{ij}$. Let $\xi \in H^1(Y', \mathcal{O})$ be the cohomology class of (g_{ij}). Then $\zeta = F\xi$. Hence from $(\star\star)$ one gets $\zeta|_Y = F\xi|_Y = 0$. $\square$

Corollary. Suppose X is a non-compact Riemann surface and $Y \Subset Y' \subset X$ are open subsets. Then for every differential form $\omega \in \mathscr{E}^{0,1}(Y')$ there exists a function $f \in \mathscr{E}(Y)$ such that

$$\frac{\partial f}{\partial \bar{z}} \mathrm{d}z = \omega|_Y.$$

Proof. By Theorem (1.3) the problem has a solution locally i.e. there exist an open covering $\mathcal{U} = (U_i)_{i \in I}$ of Y' and functions $f_i \in \mathscr{E}(U_i)$ such that

$$\frac{\partial f_i}{\partial \bar{z}} \mathrm{d}z = \omega|_{U_i}.$$

The differences $f_i - f_j$ are holomorphic on $U_i \cap U_j$ and thus define a cocycle in $Z^1(\mathcal{U}, \mathcal{O})$. By Theorem (5.2) this cocycle is cohomologous to zero on Y and thus there exist holomorphic functions $g_i \in \mathcal{O}(U_i \cap Y)$ such that

$$f_i - f_j = g_i - g_j \quad \text{on } U_i \cap U_j \cap Y.$$

Hence there exists a function $f \in \mathscr{E}(Y)$ such that for all $i \in I$:

$$f = f_i - g_i \quad \text{on } U_i \cap Y.$$

But then the function f satisfies the equation

$$\frac{\partial f}{\partial \bar{z}} \mathrm{d}z = \omega|_Y.$$

□