Exact sequences of sheaves and Dolbeault's theorem:

- sheaf hom's
- exact sequences of sheaves

$\rightsquigarrow$ exact sequence in cohomology

$\Rightarrow$ "compute" $H^1(X, \mathcal{O})$ and $H^1(X, \Omega)$

- deRham groups

Def 15.1: $\mathcal{F}, \mathcal{G}$ sheaves of abelian groups on the top space X. A sheaf homomorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a family of group homomorphisms

$$\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \qquad U \text{ open in } X$$

which are compatible with the restriction hom's:

$\forall U, V \subset X$ open with $V \subset U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\text{restr.}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d, "\text{restr.}"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"'] & \mathcal{G}(V) \end{tikzcd}$$

If all the α_U are isos, we call α an isomorphism.

Examples:

a) $\mathcal{E}, \mathcal{E}^{(1)}, \mathcal{E}^{(2)}$ sheaves of diff functions/1/2 forms on a RS X. The exterior derivative d on functions/diff forms induces sheaf hom's

$$d : \mathcal{E} \longrightarrow \mathcal{E}^{(1)}, \quad d : \mathcal{E}^{(1)} \longrightarrow \mathcal{E}^{(2)}$$

Similarly for d' and d''.

b) On a RS X, the natural inclusions

2 $\mathcal{O} \longrightarrow \mathcal{E}$, $\mathbb{C} \longrightarrow \mathcal{E}$, $\mathbb{Z} \longrightarrow \mathbb{C}$

(sheaf of locally const. functions with values in $\mathbb{C}$ resp. $\mathbb{Z}$.)

$\Omega \longrightarrow \mathcal{E}^{1,0}$

sheaf of holomorphic diff 1-forms

c) on a RS X define the sheaf hom
$ex: \mathcal{O} \longrightarrow \mathcal{O}^*$ sheaf of (mult) ab. grps
where $\mathcal{O}^*(U)$ is the mult. group of all holomorphic maps $f: U \longrightarrow \mathbb{C}^*$
for $U \subset X$ open, $f \in \mathcal{O}(U)$ let $ex_U(f) := \exp(2\pi i f)$

Def: $\mathcal{F}, \mathcal{G}$ sheaves on top space X, $\alpha: \mathcal{F} \to \mathcal{G}$ a sheaf hom. For $U \subset X$ open let

$$K(U) := \ker(\mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U)) \subset \mathcal{F}(U)$$

the family $(K(U))_{U \subset X \text{ open}}$ together with restr. homs induced from $\mathcal{F}$ forms a sheaf called kernel of α.

Not: $K = \ker \alpha$

Ex: On any RS one has

((CReq. $\Leftrightarrow \frac{\partial}{\partial \bar z} \cdots = 0$))
since we saw
f holom $\underset{(9.1)}{\Leftrightarrow} d''f = 0$

a) $\mathcal{O} = \ker(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1})$

b) $\Omega = \ker(\mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)})$ (9.16)

c) $\mathbb{Z} = \ker(\mathcal{O} \xrightarrow{ex} \mathcal{O}^*)$ (multiplicative grp!!)
$e^{2\pi i \mathbb{Z}} = 1$! so id $= 1$ not 0

9.16

9.16 Thm: Y open subset of RS.

ANNEXE

$\Rightarrow$ a) $\forall$ holom 1-form $\omega \in \Omega(Y)$ is closed

b) $\forall$ closed 1-form $\omega \in \mathcal{E}^{1,0}(Y)$ is holomorphic

Pf: ω diff 1-form of type $(1,0)$

wrt coord nbh (U, z) write $\omega = f\,dz$ for some diff. function f.

$$\Rightarrow d\omega = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z\right) \wedge dz$$

def?

$$= -\frac{\partial f}{\partial \bar z} dz \wedge d\bar z$$

So $d\omega = 0 \Leftrightarrow \frac{\partial f}{\partial \bar z} = 0$ $\square$

Consequence: u harmonic function

[twice cont diff and ie u satisfies $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$]

$\Rightarrow d'u$ is holomorphic 1-form

(For, $dd'u = d''d'u = 0$; ie $d'u$ is closed. Thm ...)

Pullback of a diff. form:

$F: X \to Y$ holom. mapping between two RS,

$\forall$ open $U \subset Y$, the map F induces hom

$$F^*: \mathcal{E}(U) \longrightarrow \mathcal{E}(F^{-1}(U))$$
$$f \longmapsto f \circ F$$

... extend to diff forms p. 68

((Rmk 15.4 Given hom $\alpha: \mathcal{F} \to \mathcal{G}$ of sheaves on top space X, one can define

②

$$\boxed{\mathcal{B}(U) := \mathrm{Im}(\mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U)) \quad \forall U \overset{\text{open}}{\subset} X}$$

This defines a presheaf $\mathcal{B}$.

In gen, not a sheaf: Axiom II (gluing) fails:

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consider sheaf hom $\boxed{\begin{array}{l} \text{ex}: \mathcal{O} \to \mathcal{O}^* \\ f \mapsto \exp(2\pi i f) \end{array}}$

on the space $\mathbb{C}^*$,

Let $U_1 = \mathbb{C}^* \setminus \mathbb{R}_-$, $U_2 = \mathbb{C}^* \setminus \mathbb{R}_+$

Define $f_k \in \mathcal{O}^*(U_k)$ (mult grp of holom functions $f: U_k \to \mathbb{C}^*$)

by $f_k(z) = z \ \forall z \in U_k$.

U_k simply conn for $k = 1, 2$

$\Rightarrow f_k \in \mathrm{Im}(\mathcal{O}(U_k) \xrightarrow{\text{ex}} \mathcal{O}^*(U_k))$ (take branch of log as preim.)

Moreover, $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

But $\nexists f \in \mathrm{Im}(\mathcal{O}(\mathbb{C}^*) \xrightarrow{\text{ex}} \mathcal{O}^*(\mathbb{C}^*))$.

with $f|_{U_k} = f_k$ for both $k = 1, 2$

since $z \mapsto z$ has no single valued log on all $\mathbb{C}^*$))

1.5.5 Exact sequences: ③

$\alpha: \mathcal{F} \to \mathcal{G}$ a sheaf hom. on top space X.

$\Rightarrow$ $\forall x \in X$ there is an induced hom. on the stalks:

$$\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

Def: A sequence of sheaf hom's $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is called **exact** if $\forall x \in X$

the sequence $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is exact

(i.e. $\ker(\beta_x) = \operatorname{Im}(\alpha_x)$)

A seq. $\mathcal{F}_1 \xrightarrow{\alpha_1} \mathcal{F}_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \mathcal{F}_n$ $n \geq 3$

of sheaf homomorphisms is called exact

if the seq. $\mathcal{F}_k \xrightarrow{\alpha_k} \mathcal{F}_{k+1} \xrightarrow{\alpha_{k+1}} \mathcal{F}_{k+2}$ is exact $\forall 1 \leq k \leq n-2$

A sheaf hom $\alpha: \mathcal{F} \to \mathcal{G}$ is called **monomorphism**

if $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact.

and **epimorphism** if $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow 0$ is exact.

An exact seq. of the form $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$

is called **short exact sequence (SES)**

Lemma 15.6: $\alpha : \mathcal{F} \to \mathcal{G}$ sheaf monomorphism on top space X $\Rightarrow$ $\forall$ open $U \subset X$ the mapping

$\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

Proof: Take $f \in \mathcal{F}(U)$ st $\alpha_U(f) = 0$.

We show $f = 0$; by def of monomorphism:

$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact ie $\forall x \in X$

$0 \to \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x$ is exact ie $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ inj.

$\Rightarrow$ Every $x \in U$ has open nbh $V_x \subset U$ st

$f|_{V_x} = 0$. ((germ of f at x: $\rho_x(f)$ equiv class in $\mathcal{F}_x$))

Sheaf Axiom I (locality) $\Rightarrow$ $f = 0$. □

Rmk 15.7: $\alpha : \mathcal{F} \to \mathcal{G}$ epimorphism $\not\Rightarrow$ $\forall U$ $\mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U)$ surjective

counter ex: $\exp : \mathcal{O} \to \mathcal{O}^*$

[$\forall x \in X$ $\exp : \mathcal{O}_x \to \mathcal{O}_x^*$ is surjective
since every non-vanishing function locally has a logarithm
but $\exp : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}^*(\mathbb{C}^*)$ is not surjective!]

Lemma 15.8: Suppose $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is an exact seq. of sheaves on top space X.

$\Rightarrow$ $\forall$ open $U \subset X$ the sequence

$0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ is exact.

[skip proof]

Examples! of SES's of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \quad \text{on a RS } X.$$

a) $0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$

- natural inclusion
- $\mathcal{O} = \operatorname{Ker}(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1})$
- exactness from Dolbeaults lemma 13.2

Recall: cf ANNEXE!

b) $\mathcal{L} := \operatorname{Ker}(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)})$ the sheaf of closed diff forms.

SES $0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} \mathcal{L} \to 0$

- natural inclusion
- $\operatorname{Ker}(d: \mathcal{E} \to \mathcal{L}) = \mathbb{C}$
- $d: \mathcal{E} \to \mathcal{L}$ epimorphism since locally every closed diff form has a primitive / is exact. ((cf 10.4 Vincent's talk W3

c) $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} \Omega \to 0$ holomorphic analogue))

- nat inclusion

d) $0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$

- we saw $\Omega = \operatorname{Ker}(\mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)})$ (((cf thm 9.16)
- $d: \mathcal{E}^{1,0} \to \mathcal{E}^{(2)}$ epimorphism:

idea: wrt local chart: (U, z)

$$d(f dz) = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz$$

take any 2-form: $g\, d\bar z \wedge dz$
Dolbeault's lemma: $\forall g \in \mathcal{E}(X) \quad \exists f \in \mathcal{E}(X)$ st $\frac{\partial f}{\partial \bar z} = g$

$$\rightsquigarrow \quad g\, d\bar z \wedge dz = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = \underbrace{d(f dz)}_{\in \mathcal{E}^{1,0}}$$

Thm 13.2 (special case of Dolbeault lemma)

ANNEXE

$X := \{z \in \mathbb{C} \mid |z| < R\}$ $0 < R \leq \infty$

$g \in \mathcal{E}(X)$

$\Rightarrow \exists f \in \mathcal{E}(X)$ st $\dfrac{\partial f}{\partial \bar{z}} = g$.

Corollaries (Thm 13.4 & Thm 13.5)

*) $X = \{z \in \mathbb{C} : |z| < R\}$ $0 < R \leq \infty$

$\Rightarrow H^1(X, \mathcal{O}) = 0$.

*) For the Riemann sphere $\underset{\mathbb{P}^1}{\text{}}$ $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Thm 12.7 $H^1(X, \mathbb{C}) = 0$
$H^1(X, \mathbb{Z}) = 0$

Thm 12.6: $H^1(X, \mathcal{E}) = 0$ for X RS.

15.10: Any hom $\alpha: \mathcal{F} \to \mathcal{G}$ of sheaves on a top sp X induces hom's in cohomology:

$$\alpha^0 : H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G})$$
$$\alpha^1 : H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G})$$

α^0: Since we had
$$H^0(X,\mathcal{F}) := \mathcal{F}(X)$$
$$H^0(X,\mathcal{G}) := \mathcal{G}(X)$$

the hom α^0 is just $\alpha_X : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$

α^1: We construct α^1: $\mathcal{U} = (U_i)_{i \in I}$ an open covering of X.

$$\alpha_{\mathcal{U}} : C^1(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{G})$$
$$\xi = (f_{ij}) \longmapsto \alpha_{\mathcal{U}}(\xi) := (\alpha(f_{ij}))$$

Recall $C^1(\mathcal{U},\mathcal{F}) := \prod_{i,j \in I^{q+1}} \mathcal{F}(U_i \cap U_j)$

check:

- cocycles get mapped to cocycles (cf cocycle relation)
- coboundaries get mapped to coboundaries

$\Rightarrow$ induces a homomorphism

$$\overline{\alpha}_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(\mathcal{U}, \mathcal{G})$$

and we take α^1 as the hom induced by all the

$$\overline{\alpha}_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(\mathcal{U}, \mathcal{G}) \qquad \forall \mathcal{U} \in \mathfrak{U}.$$

15.11 The connecting hom:

SES: $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$
of sheaves
on top space X

A connecting hom $\delta^*: H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F})$ is defined as follows:

$h \in H^0(X, \mathcal{H}) = \mathcal{H}(X)$

by def of exactness of $\mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ all the $\beta_x: \mathcal{G}_x \longrightarrow \mathcal{H}_x$ are surjective

$\Rightarrow \exists$ open covering $\mathcal{U} = (U_i)_{i \in I}$ of X and a cochain $(g_i) \in C^0(\mathcal{U}, \mathcal{G})$ st $\beta(g_i) = h|_{U_i}$ $\forall i \in I$.

$\Rightarrow \beta(g_j - g_i) = 0$ on $U_i \cap U_j$.

lemma 15.8 $\Rightarrow$ $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ is exact.

in part: $\ker(\beta_U) = \mathrm{Im}(\alpha_U)$ so $\exists f_{ij} \in \mathcal{F}(U_i \cap U_j)$ st

$\alpha(f_{ij}) = g_j - g_i$

Claim: $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$

on $U_i \cap U_j \cap U_k$: $\alpha(f_{ij} + f_{jk} - f_{ik}) = \underbrace{(g_j - g_i) + (g_k - g_j) - (g_k - g_i)}_{= 0}$

Since lemma 15.6 $\Rightarrow$ α_U inj: $f_{ik} = f_{ij} + f_{jk}$ cocycle relation. ✓

And now define $\delta^* h \in H^1(X, \mathcal{F})$ to be the cohomology class represented by (f_{ij}).

(Rmk: indep of the choices...

Theorem 15.12: X top. space, $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ a SES of sheaves on X.

$\Rightarrow$ the induced sequence

((NB: normal exact seq of groups not sheaves

$$0 \to H^0(X,\mathcal{F}) \xrightarrow{\alpha^0} H^0(X,\mathcal{G}) \xrightarrow{\beta^0} H^0(X,\mathcal{H}) \xrightarrow{\delta^*} H^1(X,\mathcal{F}) \xrightarrow{\alpha^1} H^1(X,\mathcal{G}) \xrightarrow{\beta^1} H^1(X,\mathcal{H})$$

δ^* conn. hom.

in cohomology is exact.

(Pf skipped)

Thm 15.13: $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ SES of sheaves on top space X, st $H^1(X,\mathcal{G}) = 0$.

$$\Rightarrow \quad H^1(X,\mathcal{F}) \cong \mathcal{H}(X) / \beta^0 \mathcal{G}(X)$$

Pf: Previous thm $\Rightarrow$

$$\underset{H^0(X,\mathcal{G})}{\overset{\shortparallel}{\mathcal{G}(X)}} \xrightarrow{\beta^0} \underset{H^0(X,\mathcal{H})}{\overset{\shortparallel}{\mathcal{H}(X)}} \xrightarrow{\delta^*} H^1(X,\mathcal{F}) \to 0$$

is exact. $\Rightarrow \delta^*$ surj.

1st isom thm: $H^1(X,\mathcal{F}) \cong \mathcal{H}(X) / \underset{\mathrm{Im}(\beta^0)}{\overset{\shortparallel}{\ker(\delta^*)}} = \mathcal{H}(X) / \beta^0 \mathcal{G}(X)$ $\square$

15.14: Dolbeault's theorem:

X RS. $\Rightarrow$ there are isos

$$H^1(X,\mathcal{O}) \cong \mathcal{E}^{0,1}(X) / d'' \mathcal{E}(X)$$

$$H^1(X,\Omega) \cong \mathcal{E}^{(2)}(X) / d\, \mathcal{E}^{1,0}(X)$$

Pf: Ex 15.9 a) SES $0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$

And by thm 12.6 $H^1(X, \mathcal{E}) = 0$

Apply previous thm.

Ex 15.9 d) SES $0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$

$H^1(X, \mathcal{E}^{1,0}) = 0$

15.15 The deRham groups: X RS.

1st deRahm group

$$Rh^1(X) := \frac{\ker\left(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X)\right)}{\operatorname{Im}\left(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X)\right)}$$

closed 1-forms on X ~> Notat: $\mathcal{L}$

exact 1-forms on X

We call two diff forms which are equiv in $Rh^1(X)$ (ie differ by an exact form) cohomologous.

(Rmk 1) $Rh^1(X) = 0 \iff$ Every closed 1 form $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive)

2) X simply conn. $\Longrightarrow$ thm 10.7 $Rh^1(X) = 0$ (cf W3 Vincent)

deRham's theorem: X RS.

$\Longrightarrow H^1(X, \mathbb{C}) \cong Rh^1(X)$

sheaf of closed diff forms

Pf: Ex 15.9 b) SES $0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} \mathcal{L} \to 0$

and apply thm 15.13: $H^1(X, \mathbb{C}) \cong \mathcal{L} / d\mathcal{E}(X)$ □