Riemann Surfaces L7 03.04.23

§ 16 The Riemann-Roch Theorem

The Riemann-Roch theorem tells us roughly how many linearly independent meromorphic functions there are having certain restrictions on their poles.

16.1 Divisors:

First let X be a RS. A **divisor on X** is a map $D: X \longrightarrow \mathbb{Z}$ s.t. $\forall K \subset X$ compact, there are only finitely many $x \in K$ s.t. $D(x) \neq 0$.

The set of divisors $Div(X)$ forms an abelian group under addition $D, D' \in Div(X)$:

$$(D + D')(x) = D(x) + D'(x).$$

Also, we have a partial order: $D \leq D'$ $:\Leftrightarrow$ $D(x) \leq D'(x)$ $\forall x \in X$.

16.2 Divisors of meromorphic functions and 1-forms

Sps X RS and $Y \subset X$ open. For $f \in \mathcal{M}(Y)$, $a \in Y$ define:

$$\mathrm{ord}_a(f) := \begin{cases} 0 & f \text{ hol. at } a \text{ (non-zero)} \\ \infty & f \equiv 0 \text{ in a nbhd of } a. \\ k & f \text{ has zero of order } k \text{ at } a \\ -k & f \text{ has pole of order } k \text{ at } a \end{cases}$$

Thus $\forall f \in \mathcal{M}(X) \setminus \{0\}$ $\quad x \longmapsto \operatorname{ord}_x(f)$

is a divisor on X, called <u>divisor of f</u>, denoted (f).

The function f is said to be a 'multiple' of a Div. D if $(f) \geq D$. Note f hol. $\iff$ $(f) \geq 0$.

For a meromorphic one form $\omega \in \mathcal{M}^{(1)}(X)$ one can find a chart around a, (U, z). Then on $U \cap Y$ one may write $\omega = f\,dz$

with $f \in \mathcal{M}(U \cap Y)$. Set $\operatorname{ord}_a(\omega) := \operatorname{ord}_a(f)$

This is indep of choice of chart.

For 1-forms $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ the map

$x \longmapsto \operatorname{ord}_x(\omega)$ is again a divisor on X

denoted by (ω).

Let $f, g \in \mathcal{M}(X) \setminus \{0\}$ and $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$:

Then

$$(fg) = (f) + (g) \qquad (1/f) = -(f)$$

$$(f\omega) = (f) + (\omega).$$

A $D \in \operatorname{Div}(X)$ is called <u>principal divisor</u> if $\exists f \in \mathcal{M}(X) \setminus \{0\}$ s.t. $D = (f)$. Two divisors

D, D' are said to be <u>equivalent</u> if $D - D'$ is a principal divisors.

A <u>canonical divisor</u> is a divisor (ω) of a mero. 1-form (not zero). Any two $(\omega_1), (\omega_2)$ ~~patt~~ $\omega_1, \omega_2 \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ are eq. since ~~locally~~.

$\exists f \in \mathcal{M}(X) \setminus \{0\}$ s.t. $\omega_1 = f\omega_2$

$$\Rightarrow (\omega_1) = (f) + (\omega_2) \Rightarrow (\omega_1) - (\omega_2) = (f).$$

<u>16.3 The degree of a Divisor</u>: Sps now X compact RS. Then for every $D \in Div(X)$ only fin. many $x \in X$ s.t. $D(x) \neq 0$. So:

$$\deg : Div(X) \longrightarrow \mathbb{Z}$$
$$D \longmapsto \sum_{x \in X} D(x).$$

is called <u>degree of D</u>. This is a group homomorphism $\left(\deg(D + D') = \sum_{x \in X} (D + D')(x) = \sum_{x \in X} D'(x) + \sum_{x \in X} D(x) = \deg D + \deg D' \right)$.

Note $\deg (f) = 0$ since a mero. function on a compact RS has as many poles as zeros. This is Cor 4.25, which we did not see in the course.

This implies in particular that equivalent divisors have the same degree.

16.4 The sheaves $\mathcal{O}_D$: Sps D is a divisor on a RS X. For any open $U \subset X$ define

$$\mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) \mid \operatorname{ord}_x(f) \geq -D(x) \ \forall x \in U \}.$$

to be the set of functions on U, which are multiples of $-D$. Together with restriction maps $\mathcal{O}_D$ is a sheaf. If $D = 0$ then $\mathcal{O}_D = \mathcal{O}$ the sheaf of hol. functions.

$D, D' \in \operatorname{Div}(X)$ are equivalent $\Rightarrow$ $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ isomorphic.

Let $\psi \in \mathcal{M}(X) \setminus \{0\}$ s.t. $D - D' = (\psi)$

Then $\mathcal{O}_D \longrightarrow \mathcal{O}_{D'}$, $f \longmapsto \psi f$

is a sheaf isomorphism.

16.5 Theorem: Sps X comp. RS and $D \in \operatorname{Div}(X)$ with $\deg D < 0$. Then $H^0(X, \mathcal{O}_D) = 0$.

Proof: Sps $\exists f \in H^0(X, \mathcal{O}_D) \cong \mathcal{O}_D(X)$ s.t. $f \neq 0$. Then $(f) \geq -D$ and thus:

$$\deg(f) \geq -\deg D > 0$$

$\lightning$ $\deg(f) = 0$. $\square$

16.6 The skyscraper Sheaf $\mathbb{C}_p$

Sps $p \in X$. Define the sheaf

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & p \in U \\ 0 & p \notin U \end{cases}$$

with restriction maps. Then:

(i) $H^0(X, \mathbb{C}_p) \cong \mathbb{C}_p(X) \cong \mathbb{C}$

ii) $H^1(X, \mathbb{C}_p) = 0$.

Proof: Consider a cohomology class $\xi \in H^1(X, \mathbb{C}_p)$ represented by a cocyle in $H^1(\mathcal{U}, \mathbb{C}_p)$.

Take a refinement $\mathcal{B} = (V_\alpha)_{\alpha \in A}$ s.t. P in only in one V_α. Then $Z^1(\mathcal{B}, \mathbb{C}_p) = 0$

$\Rightarrow \xi = 0$. $\square$

HALF

16.7 : Now sps D divisor on X. Let $P \in X$. ~~Denote~~ by . Denote also by P the div $P(x) = \begin{cases} 1 & x = P \\ 0 & x \neq P \end{cases}$.

Then $D \leq D + P$ and there is a nat. incl. map $\mathcal{O}_D \longrightarrow \mathcal{O}_{D+P}$

$$\mathcal{O}_D(U) = \{f \in M(U) \mid \deg(f) \geq -D\} \hookrightarrow \mathcal{O}_{D+P}(U) = \{f \mid \deg f \geq -D-P\}$$

Let (V, z) be ~~a~~ coord. on X around P s.t. $z(P) = 0$

Define $\quad \beta : \mathcal{O}_{D+P} \longrightarrow \mathbb{C}_P$.

sheaf-hom as follows: $U \subset X$ open. If $P \notin U$ ~~$B_U = 0$~~ β_U is the zero hom.

If $P \in U$ $\quad f \in \mathcal{O}_{D+P}(U)$ then f admits a Laurent series about P wrt loc. coord z:

$$f = \sum_{n=-k-1}^{\infty} c_n z^n.$$

where $k = D(P)$. Set $\beta_U(f) = c_{-k-1} \in \mathbb{C} = \mathbb{C}_P(U)$

β is a sheaf epimorphism ($\forall w \in \mathbb{C}$ we have a Laurent Exp with $c_{-k-1} = w$.)

and ths:

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P \longrightarrow 0$$

is a short exact seq. and by (15.12) we get.
(SES sheaf hom $\Rightarrow$ LES in cohom)

an exact seq:

$$\begin{array}{l} 0 \longrightarrow H^0(X, \mathcal{O}_D) \longrightarrow H^0(X, \mathcal{O}_{D+P}) \longrightarrow \overset{H^0(X,\mathbb{C}_P)}{\overset{\|}{\mathbb{C}}} \\ \longrightarrow H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}_{D+P}) \longrightarrow \underset{H^1(X,\mathbb{C}_P)}{\underset{\|}{0}} \end{array} \quad (*)$$

<u>16.8 Corollary</u> Let $D \leq D'$ be divisors on a compact RS X. Then the inclusion $\mathcal{O}_D \longrightarrow \mathcal{O}_{D'}$ induces an epimorphism

$$H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}_{D'}) \longrightarrow 0.$$

<u>Proof:</u> If $D' = D + P$ where P is the $P(x) = \begin{cases} 1 & x = p \\ 0 & x \neq p \end{cases}$
then we just showed this in 16.7.

In general $D' = D + P_1 + \dots + P_n$ ($P_i \in X$) and we apply 16.7 inductively. $\square$

16.9 The Riemann - Roch Theorem: Sps D is a divisor on a compact RS X of genus g.

Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dim. VS and

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

recall
$\boxed{g := \dim H^1(X, \mathcal{O})}$

Proof: a) $D = 0$. Then $\mathcal{O}_D = \mathcal{O}$ and

$H^0(X, \mathcal{O}) = \mathcal{O}(X)$ consists of const. functions

$\Rightarrow \dim H^0(X, \mathcal{O}) = 1.$

and $H^1(X, \mathcal{O}) = g$ by def. ✓

f hol. on comp. RS X
Comp $\Rightarrow \exists p_0 \in X$ $|f(p_0)|$ max
claim $f(\phi^{-1}(z))$ hol. on unit disk max at $\phi(p_0)$.
Max modulus principle $\Rightarrow$ const.

b) Sps D divisor, $P \in X$ and $D' = D + P$. Sps the result holds for one of D and D'.

The exact cohom. seq. (*) can be split in two by defining: $V := \operatorname{Im}(H^0(X, \mathcal{O}_{D'}) \to \mathbb{C})$

$W := \mathbb{C}/V.$

to get:

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to V \to 0$$

$$0 \to W \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0$$

are exact and $\dim W = 1 - \dim V$

all VS involved are fin. dim. $= \deg D' - \deg D$.

we get:

$$\dim H^0(X, \mathcal{O}_{D'}) = \dim H^0(X, \mathcal{O}_D) + \dim V$$

$$\dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}_{D'}) + \dim W$$

if we add we get:

$$\dim H^0(X, \mathcal{O}_{D'}) - \dim H^1(X, \mathcal{O}_{D'}) - \deg D'$$

$$= \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D$$

so RR holds for one $\Rightarrow$ RR holds for the other.

c) An arbitrary divisor D may be written $D = P_1 + \dots + P_m - P_{m+1} - \dots - P_n$

$P_j \in X$. ~~Take $D = 0$ and $D' = D +$~~

Start with the 0-divisor and then b) then by induction it holds for general D $\square$

16.10 Index of Speciality

One calls $i(D) := \dim H^1(X, \mathcal{O}_D)$ the index of speciality of ~~D~~. Thus Riemann - Roch may be ~~writte~~ written:

$$\dim H^0(X, \mathcal{O}_D) = 1 - g + \deg D + i(D).$$

[In sec. 17.16 we show $i(D) = 0$ whenever $\deg D \geq 2g-2$]

By 16.5 $i(D) = g - 1 - \deg D$ if $\deg D < 0$

16.11 Thm: Sps X compact RS of genus g and $a \in X$. Then $\exists$ non - const. ~~hol.~~ meromorphic function f on X which has a pole of order $\leq g+1$ at a and is otherwise holomorphic

Proof: $D : X \longrightarrow \mathbb{Z}$ be the divisor $D(a) = g+1$, $D(x) = 0 \ \forall x \neq a$

Then by RR:

$$\dim H^0(X, \mathcal{O}_D) \geq 1 - g + \deg D = 2$$

Thus $\exists$ non - const $f \in H^0(X, \mathcal{O}_D)$ and this function fulfills the requirements. $\square$.

16.12 Corollary: Sps X RS of genus g. Then there exist a holomorphic covering map $f: X \to \mathbb{P}^1$ with at most $g+1$ sheets.

[→ preimage of comp. set is comp.]

Fact 4.24: X, Y RS $f: X \to Y$ proper non-const hol. map. Then $\exists\, n \in \mathbb{N}$ s.t. f takes every value $c \in Y$ counting multiplicities n times

Proof: We take the function from (16.11), by Fact 4.24 this is a covering mapping since, the value ∞ is attained $\leq g+1$. $\square$

16.13 Corollary: Every simply connected RS of genus zero is isomorphic to ~~S^1~~ $\mathbb{P}^1$.

Proof: By 16.12 there is a 1-sheeted covering of $\mathbb{P}^1$. This has to be a biholomorphism. $\square$

Ex 16.1 : D divisor on $\mathbb{P}^1$. Prove

a) $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = \max(0, 1 + \deg D)$

Proof: Assume $\deg D < 0$ then 16.5 $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = 0$

Assume $\deg D = 0$: $\mathcal{O}_D = \mathcal{O}$ and like before $H^0(\mathbb{P}^1, \mathcal{O}_D)$ consists of const. functions

$\Rightarrow \dim H^0(\mathbb{P}^1, \mathcal{O}_0)$

Now by induction for $\deg D > 0$: (Base $= 0$)

Sps the result true for $\deg D = n$.

Then assume $\deg D = n+1$. then:

$$\mathcal{O}_D(X) = \left\{ f \in M(X) \mid (f) \geq -\deg D \right\}$$

$$= \underbrace{\left\{ f \in M(X) \mid (f) \geq -n \right\}}_{\dim\ n+1} \oplus \left\{ f \in M(X) \mid (f) = -n-1 \right\}$$

$$\shortparallel$$

$$\langle g \rangle$$

$$g: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
$$x \longmapsto \frac{1}{(x)^{n+1}}$$

$\Rightarrow \dim \mathcal{O}_D(X) = n+2$

$= \deg D + 1$ ✓