Serre Duality & Riemann-Hurwitz formula

Def (Res) let X be a compact Riemann surface & let $\text{Res}: H^1(X, \Omega) \to \mathbb{C}$ (1st cohomology group). Recall that by Dolbeault, we have an iso $H^1(X,\Omega) \cong \mathcal{E}^{(2)}(X) / d\mathcal{E}^{1,0}(X)$.

For $\xi \in H^1(X,\Omega)$ let $\omega \in \mathcal{E}^{(2)}(X)$ be a representative of ξ via this iso.

Set the linear form

$$\boxed{\text{Res}(\xi) := \frac{1}{2\pi i} \iint_X \omega}$$

which is independent of the choice of representative ω because for $\eta \in \mathcal{E}^{(1)}(X)$ we have $\iint_X d\eta = 0$ and since any other representative differs from ω by $d\eta$ for some $\eta \in \mathcal{E}^{(1)}(X)$, the claim follows.

Def (Mittag-Leffler Distribution) let X be a Riemann surface, $\mathcal{M}^{(1)}$ the sheaf of meromorphic 1-forms on X & $\mathcal{U} = (U_i)_{i \in I}$ an open covering of X. A cochain $\mu = (\omega_i)_{i \in I} \in C^0(\mathcal{U}, \mathcal{M}^{(1)})$ is called a Mittag-Leffler distribution if the differences $\omega_j - \omega_i$ are holomorphic on $U_i \cap U_j$, i.e. $\delta\mu \in Z^1(\mathcal{U}, \Omega)$. Now we can denote by $[\delta\mu] \in H^1(X,\Omega)$ its cohomology class.

For $a \in X$, define the residue of μ as above at the point a as follows: choose $i \in I$ s.t. $a \in U_i$ & set

$$\boxed{\text{Res}_a(\mu) := \text{Res}_a(\omega_i).} \quad (2)$$

which is independent of the choice of $i \in I$ since if $a \in U_i \cap U_j$, $\omega_i - \omega_j$ is holomorphic & thus does not have a c_{-1} coefficient $\Rightarrow c_{-1}^{\omega_i} = c_{-1}^{\omega_j}$.

For X compact, $\text{Res}_a(\mu) \neq 0$ only for finitely many points a (since $\text{Res}_\cdot(\mu): X \to \mathbb{C}$ must have compact image & choosing a cover (Spt. 4) of $\mathbb{R}$, there can only be finitely many $a \in X$ s.t. $\text{Res}_a(\mu) \neq 0$). Define

$$\boxed{\text{Res}(\mu) := \sum_{a \in X} \text{Res}_a(\mu)}$$

Goal: $\exists$ relation between this residue & the linear form Res defined in the beginning.

Theorem 17.3 With the same notation, we have $\mathrm{Res}(\mu) = \mathrm{Res}([\delta\mu])$.

proof omitted, rough idea: construct iso $H^1(X,\Omega) \cong \mathcal{E}^{(2)}(X)/d\mathcal{E}^{1,0}(X)$ explicitly. 13'

Def (sheaves Ω_D) X compact Riemann surface. $\forall D \in \mathrm{Div}(X)$, denote Ω_D = sheaf of meromorphic 1-forms which are multiples of $-D$. Hence, $\forall U \subseteq X$ open, $\Omega_D(U) = \{\omega \in \underbrace{\mathcal{M}^{(1)}(U)}_{\text{meromorphic 1-forms on } U} \mid \mathrm{ord}_x(\omega) \geq -D(x) \ \forall x \in U\}$.

The restriction homomorphisms are given by the natural restriction maps.

Note: $\Omega_0 = \Omega$.

Let $\omega \in \mathcal{M}^{(1)}(X)$ be a non-trivial meromorphic 1-form on X, e.g. $\omega = df$ for $f \in \mathcal{M}(X)$ a non-cst. meromorphic fct. Let K be the divisor of ω. Then $\forall D \in \mathrm{Div}(X)$, multiplication by ω induces a sheaf

iso $\quad \mathcal{O}_{D+K} \xrightarrow{\sim} \Omega_D,\ f \mapsto f\omega \qquad [16.2: (fg) = (f) + (g)]$

$\mathcal{O}_D(U) := \{f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x)\ \forall x \in U\}$

Lemma 17.4 $\quad \exists k_0 \in \mathbb{Z}$ s.t. $\dim H^0(X,\Omega_D) \geq \deg D + k_0 \quad \forall D \in \mathrm{Div}(X)$.

proof Let ω, K be as above & g the genus of X. Set $k_0 := 1 - g + \deg K$. By Riemann-Roch,

$\dim H^0(X,\Omega_D) = \dim H^0(X,\mathcal{O}_{D+K}) = \dim H^1(X,\mathcal{O}_{D+K}) + 1 - g + \deg(D+K) \geq \deg D + k_0.$ □

Def (dual pairing) X compact Riemann surface, $D \in \mathrm{Div}(X)$. The product $\Omega_{-D} \times \mathcal{O}_D \to \Omega, (\omega, f) \mapsto \omega f$

induces a mapping $H^0(X,\Omega_{-D}) \times H^1(X,\mathcal{O}_D) \to H^1(X,\Omega)$.

Composing with $\mathrm{Res}: H^1(X,\Omega) \to \mathbb{C}$ produces a bilinear mapping

$$\langle\,,\rangle : H^0(X,\Omega_{-D}) \times H^1(X,\mathcal{O}_D) \to \mathbb{C},\ \langle \omega, \xi \rangle := \mathrm{Res}(\omega\xi)$$

This in turn induces a linear mapping $\iota_D : H^0(X,\Omega_{-D}) \to H^1(X,\mathcal{O}_D)^*,\ \omega \mapsto (\xi \mapsto \langle \omega, \xi \rangle)$

We'll see that the Serre Duality thm asserts that $\langle\,,\rangle$ is a dual pairing, i.e. ι_D is an iso.

We can already show injectivity:

Theorem 17.6 ι_D is injective.

proof we want to show: $\forall\, 0 \neq \omega \in H^0(X, \Omega_{-D})\ \exists\, \xi \in H^1(X, \mathcal{O}_D)$ s.t. $\langle \omega, \xi \rangle \neq 0$:

let $a \in X$ s.t. $D(a) = 0$ & (U_0, z) a coordinate neighborhood of a with $z(a) = 0$ & $D|_{U_0} = 0$.

Write $\omega = f\,dz$ on U_0 with $f \in \mathcal{O}(U_0)$. We may assume U_0 small enough s.t. f has no zeros in $U_0 \setminus \{a\}$.

$U_1 := X \setminus \{a\}$, $\mathfrak{U} = (U_0, U_1)$. With $\eta := (f_0, f_1) \in C^0(\mathfrak{U}, \mathcal{M})$, $f_0 = (zf)^{-1}$, $f_1 = 0$, we have $\omega\eta = (\frac{dz}{z}, 0) \in C^0(\mathfrak{U}, \mathcal{M}^{(1)})$

which is a Mittag-Leffler distribution with $\mathrm{Res}(\omega\eta) = 1$.

With $\delta\eta \in Z^1(\mathfrak{U}, \mathcal{O}_D)$, we set $\xi = [\delta\eta] \in H^1(X, \mathcal{O}_D)$ its cohomology class. Using $\omega\xi = \omega \cdot [\delta\eta] = [\delta(\omega\eta)]$

we get from thm 17.3 that $\langle \omega, \xi \rangle = \mathrm{Res}(\omega\xi) = \mathrm{Res}([\delta(\omega\eta)]) = \mathrm{Res}(\omega\eta) = 1 \neq 0$. □

For $D' \leq D \in \mathrm{Div}(X)$ for X a compact Riemann surface, the following diagram commutes

$$\begin{array}{ccccc} 0 \to & H^1(X, \mathcal{O}_D)^* & \xrightarrow{i_{D'}^D} & H^1(X, \mathcal{O}_{D'})^* \\ & \iota_D \uparrow & & \uparrow \iota_{D'} \\ 0 \to & H^0(X, \Omega_{-D}) & \longrightarrow & H^0(X, \Omega_{-D'}) \end{array}$$

where $i_{D'}^D$ is defined by the induced monomorphism of duals (sequence below is exact (exactness is def. on stalks))

$$0 \to H^1(X, \mathcal{O}_D)^* \xrightarrow{i_{D'}^D} H^1(X, \mathcal{O}_{D'})^*$$

via the epimorphism $H^1(X, \mathcal{O}_{D'}) \to H^1(X, \mathcal{O}_D)$ (sequence $H^1(X, \mathcal{O}_{D'}) \to H^1(X, \mathcal{O}_D) \to 0$ is exact)

induced by the inclusion $0 \to \mathcal{O}_{D'} \to \mathcal{O}_D$ by 16.8 (4)

lemma With the same notation, let $\lambda \in H^1(X, \mathcal{O}_D)^*$ and $\omega \in H^1(X, \mathcal{O}_{D'})^*$ satisfy $i_{D'}(\lambda) = \iota_{D'}(\omega)$. Then $\omega \in H^0(X, \Omega_{-D'})$ and $\lambda = \iota_D(\omega)$.

proof omitted (6)

Now let $D, B \in \mathrm{Div}(X)$, X cpt. Riemann surface. For $\psi \in H^0(X, \mathcal{O}_B)$ a meromorphic fct., the sheaf morphism

$$\mathcal{O}_{D-B} \xrightarrow{\psi} \mathcal{O}_D,\ f \mapsto \psi \cdot f$$

induces a linear mapping $H^1(X, \mathcal{O}_{D-B}) \to H^1(X, \mathcal{O}_D)$ & thus

$$H^1(X, \mathcal{O}_D)^* \to H^1(X, \mathcal{O}_{D-B})^*$$

a linear mapping which we'll denote by ψ as well.

By definition, $(\psi\lambda)(\xi) = \lambda(\psi\xi)$ for $\lambda \in H^1(X, \mathcal{O}_D)^*$, $\xi \in H^1(X, \mathcal{O}_{D-B})^*$.

With $\langle \psi\omega, \xi \rangle = \langle \omega, \psi\xi \rangle$ the diagram commutes

$$\begin{array}{ccc} H^1(X, \mathcal{O}_D)^* & \xrightarrow{\psi} & H^1(X, \mathcal{O}_{D-B})^* \\ \uparrow \iota_D & & \uparrow \iota_{D-B} \\ H^0(X, \Omega_{-D}) & \longrightarrow & H^0(X, \Omega_{-D+B}) \end{array}$$

lemma If $\psi \in H^0(X, \mathcal{O}_B)$ is $\neq 0$, then the mapping $\psi: H^1(X, \mathcal{O}_D)^* \to H^1(X, \mathcal{O}_{D-B})^*$ is injective.

proof let $A := (\psi) \geq -B$ be the divisor of ψ. The mapping $\mathcal{O}_{D-B} \xrightarrow{\psi \cdot} \mathcal{O}_D$ factors through $\mathcal{O}_{D+A}$, i.e.

$$\mathcal{O}_{D-B} \to \mathcal{O}_{D+A} \xrightarrow{\psi \cdot} \mathcal{O}_D$$

where $\mathcal{O}_{D+A} \xrightarrow{\psi \cdot} \mathcal{O}_D$ is an iso. With $H^1(X, \mathcal{O}_{D-B}) \to H^1(X, \mathcal{O}_{D+A})$, induced by the inclusion $\mathcal{O}_{D-B} \to \mathcal{O}_{D+A}$, being an epimorphism (16.8), we get that

$$H^1(X, \mathcal{O}_{D-B}) \xrightarrow{\psi \cdot} H^1(X, \mathcal{O}_D)$$

is also an epimorphism which shows that ψ is injective. □

Theorem (Serre duality) For any divisor D on a compact Riemann surface X, the mapping

$$\iota_D : H^0(X, \Omega_{-D}) \to H^1(X, \mathcal{O}_D)^*$$

is an isomorphism.

proof only surjectivity remains to be proved. let $0 \neq \lambda \in H^1(X, \mathcal{O}_D)^*$. Goal: show $\lambda \in \operatorname{Im}(\iota_D)$.

- let $P \in \operatorname{Div}(X)$ with $\deg P = 1$. For $n \in \mathbb{N}$, set $D_n := D - nP$.
  $\Lambda \subseteq H^1(X, \mathcal{O}_{D_n})^*$ is the vs of all linear forms of the form $\psi\lambda$, where $\psi \in H^0(X, \mathcal{O}_{nP})$.
  By the previous lemma, $\Lambda \cong H^0(X, \mathcal{O}_{nP})$. By Riemann-Roch, $\dim \Lambda \geq 1 - g + n$, $g :=$ genus of X.
- By lemma 17.4, $\operatorname{Im}(\iota_D) \subseteq H^1(X, \mathcal{O}_{D_n})^*$ satisfies $\dim \operatorname{Im}(\iota_{D_n}) = \dim H^0(X, \Omega_{-D_n}) \geq n + k_0 - \deg D$.
  $n > \deg D$, then $\deg D_n < 0$ & thus $H^0(X, \mathcal{O}_{D_n}) = 0$. Using Riemann-Roch, we get

$$\dim H^1(X, \mathcal{O}_D)^* = g - 1 - \deg D_n = n + (g - 1 - \deg D).$$

  Choosing n sufficiently large now yields $\dim \Lambda + \dim \operatorname{Im}(\iota_{D_n}) > \dim H^1(X, \mathcal{O}_{D_n})^*$

  $\implies \Lambda \cap \operatorname{Im}(\iota_{D_n}) \neq 0 \to \exists\, 0 \neq \psi \in H^0(X, \mathcal{O}_{nP})$, $\omega \in H^0(X, \Omega_{-D_n})$ s.t. $\psi\lambda = \iota_{D_n}(\omega)$.
- $A := (\psi)$ divisor of ψ, i.e. $1/\psi \in H^0(X, \mathcal{O}_A)$, $D' := D_n - A$. Then $\iota_{D'}(\lambda) = \frac{1}{\psi}(\psi\lambda) = \frac{1}{\psi}\iota_{D_n}(\omega) = \iota_{D'}(\frac{1}{\psi}\omega)$.
  By injectivity of ι_D, $\omega_0 := (1/\psi)\omega \in H^0(X, \Omega_{-D})$ and $\lambda = \iota_D(\omega_0)$. □

Theorem The divisor of a nowhere vanishing meromorphic 1-form ω on a compact Riemann surface of genus g satisfies $\deg(\omega) = 2g-2$.

proof Let $K := (\omega)$. By Riemann-Roch, $\dim H^0(X, \mathcal{O}_K) - \dim H^1(X, \mathcal{O}_K) = 1-g+\deg K$.

By the isomorphism ($\mathcal{O}_{D+K} \xrightarrow{\sim} \Omega_D$, $f \mapsto f\omega$) introduced in the beginning, $\Omega \cong \mathcal{O}_K$.

$\Rightarrow$ $1-g+\deg K = \dim H^0(X, \Omega) - \dim H^1(X, \Omega) = g-1$ $\quad\Rightarrow \deg K = 2g-2$. □

Def (branching order) Let X, Y be cpt. Riemann surfaces & $f: X \to Y$ a non-cst. holomorphic mapping. For $x \in X$, let $v(x, f)$ be the multiplicity[5] with which f takes the value $f(x)$ at x. The number

$$\boxed{b(f,x) := v(f,x) - 1}$$

is called the branching order of f at x. We have $b(f,x) = 0 \Leftrightarrow f$ unbranched at x.

X cpt $\Rightarrow \exists$ finitely many points $x \in X$ s.t. $b(f,x) \neq 0$. Thus,

$$\boxed{b(f) := \sum_{x \in X} b(f,x)}$$

the total branching order of f, is well-def.

Theorem (Riemann-Hurwitz formula) $f: X \to Y$ n-sheeted holomorphic covering map between Riemann surfaces X & Y with total branching order $b = b(f)$. Let $g =$ genus of X, $g' =$ genus of Y. Then

$$\boxed{g = \frac{b}{2} + n(g'-1) + 1}$$

proof Let ω be a nowhere vanishing meromorphic 1-form on Y. Then $\deg(\omega) = 2g'-2$ and $\deg(f^*\omega) = 2g-2$.

Let $x \in X$, $f(x) = y$. $\exists$ coordinate neighborhood (U, z) of x [(U', w) of y] with $z(x) = 0$ [$w(y) = 0$] s.t. we may write f wrt these coordinates as $w = z^k$ where $k = v(f,x)$.

Let $\omega = \psi(w)\,dw$ on U'. Then on U one has

$$f^*\omega = \psi(z^k)\,dz^k = k z^{k-1} \psi(z^k)\,dz.$$

$\Longrightarrow \operatorname{ord}_x(f^*\omega) = b(f,x) + v(f,x)\operatorname{ord}_y(\omega)$

With $\sum_{x\in f^{-1}(y)} v(f,x) = n$, $\forall y\in Y$ we have $\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega) = \sum_{x\in f^{-1}(y)} b(f,x) + n\operatorname{ord}_y(\omega)$.

$\Longrightarrow \deg(f^*\omega) = \sum_{x\in X} \operatorname{ord}_x(f^*\omega) = \sum_{y\in Y}\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega) = \sum_{x\in X} b(f,x) + n\sum_{y\in Y}\operatorname{ord}_y(\omega) = b(f) + n\deg(\omega)$

$\Longrightarrow 2g-2 = b + n(2g'-2)$. $\square$