APPLICATIONS OF THE RIEMANN-ROCH AND SERRE DUALITY THEOREMS AND EMBEDDINGS INTO PROJECTIVE SPACE

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Contents

1. Recap of the last two talks

Let us make a quick recap of some of the notions and results we saw in the last two talks.

We start with the definition of a divisor, its degree, and its induced sheaf of meromorphic functions.

Definition 1.1 (cf. [For81, Paragraphs 16.1-16.4])**.** *Let X be a Riemann surface.*

A divisor *on X is a map $D : X \to \mathbb{Z}$ so that for every compact subset $K \subset X$, there are only finitely many points in K at which D takes a non-zero value.*

The sheaf $\mathcal{O}_D$ induced by D is defined as

$$\mathcal{O}_D(U) := \{f \in \mathcal{M}(U) \ \mid \forall x \in U : \operatorname{ord}_x(f) \geq -D(x)\}$$

for every open subset $U \subset X$, with restriction maps being the usual restrictions of functions.

When X is compact, for a divisor D on X, we define its degree *as the number*

$$\deg(D) := \sum_{x \in X} D(x).$$

When the degree of D is negative, the zeroth cohomology group with coefficients in $\mathcal{O}_D$ is trivial.

Lemma 1.2 (cf. [For81, Theorem 16.5])**.** *Let X be a compact Riemann surface and let D be a divisor on X with $\deg(D) < 0$.*

Then,

$$H^0(X, \mathcal{O}_D) = 0.$$

Now, let X be a compact Riemann surface, and let $f \in \mathcal{M}(X)$. Then, recall that f induces a divisor (f) on X defined as

$$(f) : X \to \mathbb{Z}, \quad x \mapsto \operatorname{ord}_x(f).$$

This is called a *principal divisor*. In a similar manner, a meromorphic 1-form on X also induces a divisor on X.

Two divisors D, D' on X are said to be *equivalent* if

$$D - D' = (f)$$

for some $f \in \mathcal{M}(X)$. In this case, the map

$$\mathcal{O}_D \to \mathcal{O}_{D'}, \quad \psi \mapsto f\psi \tag{1.1}$$

is a sheaf isomorphism.

Now, for two divisors D and D' on X satisfying $D \leq D'$ in the pointwise sense, recall that the inclusion map $\mathcal{O}_D \hookrightarrow \mathcal{O}_{D'}$ induces a map in cohomology which is surjective.

Lemma 1.3 (cf. [For81, Corollary 16.8])**.** *Let X be a compact Riemann surface and let D, D' be two divisors on X with $D \leq D'$ in the pointwise sense.*

Then, the inclusion map $\mathcal{O}_D \hookrightarrow \mathcal{O}_{D'}$ induces a surjective linear map

$$H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}).$$

Next, we recall the statement of the Riemann-Roch theorem.

Theorem 1.4 (Riemann-Roch, cf. [For81, Theorem 16.9]). *Let X be a compact Riemann surface of genus $g \geq 0$ and let D be a divisor on X.*

Then, $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite-dimensional and satisfy

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) = 1 - g + \deg(D).$$

We now move towards the statement of the Serre Duality Theorem. Let X be a compact Riemann surface. Recall the short exact sequence of sheaves

$$0 \to \Omega \hookrightarrow \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)} \to 0$$

which states that a type $(1,0)$ form is holomorphic if and only if it is closed. The induced long exact sequence in cohomology implies that

$$H^1(X, \Omega) \simeq \mathscr{E}^{(2)}(X)/d(\mathscr{E}^{1,0}(X)).$$

Hence, we may define a linear map

$$\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}, \quad [\omega] \mapsto \frac{1}{2\pi i} \int_X \omega,$$

where $[\omega] \in H^1(X, \Omega) \simeq \mathscr{E}^{(2)}(X)/d(\mathscr{E}^{1,0}(X))$, and the above is well-defined by Stokes's theorem.

To state the Serre Duality Theorem, we recall the definition of the sheaf of 1-forms induced by the divisor.

Definition 1.5 (cf. [For81, Paragraph 17.4]). *Let X be a Riemann surface and let D be a divisor on X.*

We define the sheaf Ω_D by

$$\Omega_D(U) := \{\omega \in \mathcal{M}^{(1)}(U) \mid \forall x \in U : \mathrm{ord}_x(\omega) \geq -D(x)\},$$

for every $U \subset X$ open, and the restriction maps are the usual restrictions of 1-forms.

Remark 1.6 (cf. [For81, Paragraph 17.4]). Let a compact X be a Riemann surface and let D be a divisor on X. Let $\omega \neq 0$ be a meromorphic 1-form on X, and denote by K the divisor induced by ω.

Then, the map

$$\mathcal{O}_{D+K} \to \Omega_D, \quad f \mapsto f\omega \tag{1.2}$$

is a sheaf isomorphism.

With the above notions, we can now state the Serre Duality Theorem.

Theorem 1.7 (Serre duality, cf. [For81, Theorem 17.9]). *Let X be a compact Riemann surface and let D be a divisor on X.*

Then, the pairing

$$\Omega_{-D} \times \mathcal{O}_D \to \Omega, \quad (\omega, f) \mapsto f\omega$$

induces a bilinear map

$$H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \to H^1(X, \Omega)$$

for which the map resulting from the composition

$$H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \to H^1(X, \Omega) \overset{\mathrm{Res}}{\to} \mathbb{C}$$

is non-degenerate.

The following is a direct corollary of the Serre Duality Theorem.

Corollary 1.8 (cf. [For81, Paragraphs 17.9 and 17.11])**.** *Let X be a compact Riemann surface and let D be a divisor on X.*

Then,

$$H^0(X, \Omega_{-D}) \simeq H^1(X, \mathcal{O}_D)^*$$

and

$$H^0(X, \mathcal{O}_{-D}) \simeq H^1(X, \Omega_D)^*.$$

Another corollary of the Riemann-Roch and the Serre Duality Theorems is the computation of the degree of a canonical divisor in terms of the genus of a Riemann surface.

Proposition 1.9 (cf. [For81, Theorem 17.12])**.** *Let X be a compact Riemann surface of genus $g \geq 0$ and let $\omega \neq 0$ be a meromorphic 1-form on X. Denote by K the divisor induced by ω.*

Then,

$$\deg(\omega) := \deg(K) = 2g - 2.$$

2. Some simple applications of the Riemann-Roch and Serre Duality theorems

In this section, we discuss some simple applications of the Riemann-Roch and the Serre Duality Theorems.

We start with the computation of the first cohomology group with coefficients in the sheaf of meromorphic functions induced by a divisor of "large" degree.

Theorem 2.1 (cf. [For81, Theorem 17.16])**.** *Let X be a compact Riemann surface of genus $g \geq 0$ and let D be a divisor on X with $\deg(D) \geq 2g-1$.*

Then,

$$H^1(X, \mathcal{O}_D) = 0.$$

Proof. Let $\omega \neq 0$ be a meromorphic 1-form on X, and denote by K the induced divisor. By (1.2), the sheaves Ω_{-D} and $\mathcal{O}_{K-D}$ are isomorphic, hence

$$H^0(X, \Omega_{-D}) \simeq H^0(X, \mathcal{O}_{K-D}).$$

By the Serre Duality Theorem (cf. Corollary 1.8) and by the above, we get

$$H^1(X, \mathcal{O}_D)^* \simeq H^0(X, \Omega_{-D}) \simeq H^0(X, \mathcal{O}_{K-D}).$$

Now, by assumption,

$$\deg(D) \geq 2g-1.$$

By Proposition 1.9,

$$\deg(K) = 2g-2,$$

and thus

$$\deg(K-D) = \deg(K) - \deg(D) < 0.$$

Hence, by Lemma 1.2, we have

$$H^0(X, \mathcal{O}_{K-D}) = 0,$$

and so $H^1(X, \mathcal{O}_D) = 0$. □

Corollary 2.2. *Let X be a compact Riemann surface. Denote by $\mathcal{M}$ the sheaf of meromorphic functions on X.*

Then,

$$H^1(X, \mathcal{M}) = 0.$$

Proof. Since X is compact, it suffices to work only with finite covers in the Čech cohomology group $H^1(X, \mathcal{M})$. Let $\mathfrak{U} = (U_i)_{i \in I}$ be a finite open cover of X and let

$$(f_{ij}) \in Z^1(\mathfrak{U}; \mathcal{M})$$

be a Čech 1-cocycle. Then, each f_{ij} is a meromorphic function on $U_i \cap U_j$, with $f_{ij} = -f_{ji}$. Hence, the number of poles of the f_{ij}'s are finite since the cover is finite.

Let D be a divisor on X so that $\deg(D) \geq 2g-1$ and so that

$$f_{ij} \in Z^1(\mathfrak{U}; \mathcal{O}_D).$$

By Theorem 2.1, $(f_{ij})_{i,j}$ is cohomologous to zero in $Z^1(\mathfrak{U}, \mathcal{O}_D)$, and hence also in $Z^1(\mathfrak{U}, \mathcal{M})$. Thus,

$$H^1(\mathfrak{U}, \mathcal{M}) = 0,$$

and so as $\mathfrak{U}$ was arbitrary, it follows that

$$H^1(X, \mathcal{M}) = 0.$$

□

Let us now see another application of these results, where we bound the dimension of $H^0(X, \mathcal{O}_D)$ for a divisor D on a compact Riemann surface X in terms of $\deg(D)$ and the genus of X.

Proposition 2.3 (cf. [For81, Exercise 17.4])**.** *Let X be a compact Riemann surface of genus $g \geq 0$ and let D be a divisor on X.*

Then, we have:

(1) $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) = 0$ *if* $\deg(D) \leq -1$.

(2) $0 \leq \dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) \leq 1 + \deg(D)$ *if* $0 \leq \deg(D) \leq g - 1$.

(3) $1 - g + \deg(D) \leq \dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) \leq g$ *if* $g - 1 \leq \deg(D) \leq 2g - 1$.

(4) $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) = 1 - g + \deg(D)$ *if* $\deg(D) \geq 2g - 1$.

Proof. Recall the Riemann-Roch theorem (cf. Theorem 1.4):

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) = 1 - g + \deg(D) + \dim_{\mathbb{C}} H^1(X, \mathcal{O}_D). \tag{2.1}$$

Firstly, if $\deg(D) \leq -1$, then $H^0(X, \mathcal{O}_D) = 0$ by Lemma 1.2.

Secondly, if $\deg(D) \geq 2g - 1$, then $H^1(X, \mathcal{O}_D) = 0$ by Theorem 2.1, so

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) = 1 - g + \deg(D)$$

by (2.1).

Thirdly, assume that $0 \leq \deg(D) \leq g - 1$. We show that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) \leq 1 + \deg(D).$$

Without loss of generality, we may assume that $H^0(X, \mathcal{O}_D) \neq 0$. Observe that we may assume that $D \geq 0$ in the pointwise sense. Indeed, if this is not the case, then we may replace D by $D' := D + (f)$ for some non-zero $f \in H^0(X, \mathcal{O}_D)$ (and also, by construction, $D' \geq 0$ in the pointwise sense) and study $H^0(X, \mathcal{O}_{D'})$, which is isomorphic to $H^0(X, \mathcal{O}_D)$ by (1.1). By construction, $D' \geq 0$ in the pointwise sense.

Now, by (2.1), it suffices to show that

$$\dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) \leq g.$$

However, since $D \geq 0$, by Lemma 1.3, we have

$$\dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) \leq \dim_{\mathbb{C}} H^1(X, \mathcal{O}_0) = \dim_C H^1(X, \mathcal{O}) = g,$$

where the last equality is the definition of the genus of X, hence $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) \leq g$.

Lastly, assume that $g - 1 \leq \deg(D) \leq 2g - 1$. We show that

$$1 - g + \deg(D) \leq \dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) \leq g.$$

By (2.1), we have

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) = 1 - g + \deg(D) + \dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) \geq 1 - g + \deg(D).$$

To show the other inequality, let $\omega \neq 0$ be a meromorphic 1-form on X. Denote by K the divisor induced by ω. By Proposition 1.9, we have $\deg(K) = 2g - 2$. By (1.2), we have a sheaf isomorphism

$$\Omega_{-D} \simeq \mathcal{O}_{K-D},$$

hence, by the Serre Duality Theorem (cf. Corollary 1.8), we obtain

$$H^1(X, \mathcal{O}_D)^* \simeq H^0(X, \Omega_{-D}) \simeq H^0(X, \mathcal{O}_{K-D}).$$

Since

$$\deg(K - D) = \deg(K) - \deg(D) = 2g - 2 - \deg(D),$$

the assumptions on $\deg(D)$ imply that $0 \leq \deg(K-D) \leq g-1$, so the second part of this proposition implies that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_{K-D}) \leq 1 + \deg(K-D).$$

Hence, by (2.1), we obtain

$$\begin{aligned}
\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) &= 1 - g + \deg(D) + \dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) \\
&= 1 - g + \deg(D) + \dim_{\mathbb{C}} H^0(X, \mathcal{O}_{K-D}) \\
&\leq 1 - g + \deg(D) + 1 + 2g - 2 - \deg(D) \\
&= g.
\end{aligned}$$

□

3. Embeddings into complex projective space

In this section, we show that every compact Riemann surface admits a holomorphic embedding into some complex projective space.

Let us first define the complex projective space of arbitrary dimension.

Definition 3.1 (Complex projective space)**.** *Let $n \geq 1$ be an integer.*

On $\mathbb{C}^{n+1} \setminus \{0\}$, we consider the relation $\sim$ defined as

$$(z_0, \cdots, z_n) \sim (z_0', \cdots, z_n') \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : (z_0, \cdots, z_n) = \lambda(z_0', \cdots, z_n'),$$

for $(z_0, \cdots, z_n), (z_0,' \cdots, z_n') \in \mathbb{C}^{n+1}$. It is straightforward to check that $\sim$ is an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$.

*We define the n-*dimensional complex projective space *as the quotient*

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim,$$

and we endow this space with the quotient topology.

We denote the equivalence class of $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ in $\mathbb{CP}^n$ as

$$[z_0 : \cdots : z_n] \in \mathbb{CP}^n.$$

For $n \in \mathbb{N}$, consider the cover of $\mathbb{CP}^n$ by the subsets

$$U_j := \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_j \neq 0\}$$

for $0 \leq j \leq n$. For such j, define the map

$$\varphi_j : U_j \to \mathbb{C}^n, \quad \varphi_j([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j} \right). \tag{3.1}$$

It is not difficult to check that each φ_j is a well-defined homeomorphism from U_j to $\mathbb{C}^n$, with inverse explicitly given by

$$\varphi_j^{-1} : \mathbb{C}^n \to U_j, \quad (x_1, \cdots, x_n) \mapsto [x_1 : x_2 : \cdots : x_{j-1} : 1 : x_j : x_{j+1} : \cdots : x_n].$$

Moreover, for $0 \leq j < k \leq n$, we have

$$(\varphi_j \circ \varphi_k^{-1})(x_1, \cdots, x_n) = \left(\frac{x_1}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_{k-1}}{x_j}, \frac{1}{x_j}, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_n}{x_j} \right)$$

for all $(x_1, \cdots, x_n) \in \mathbb{C}^n$ with $\varphi_k(x_1, \cdots, x_n) \in U_j$, and hence $\mathbb{CP}^n$ has the structure of an n-dimensional complex manifold.

Now, we can make precise the notion of "holomorphic embedding" of a Riemann surface in a complex projective space.

Definition 3.2 (cf. [For81, Paragraph 17.20])**.** *Let X be a compact Riemann surface, let $n \in \mathbb{N}$, and let $F : X \to \mathbb{CP}^n$ be a map. Consider the cover of $\mathbb{CP}^n$ with coordinate charts (U_j, φ_j) as above, for $0 \leq j \leq n$.*

We say that F is holomorphic if for every $0 \leq j \leq n$, the components of the map

$$\varphi_j \circ F : F^{-1}(U_j) \to \mathbb{C}^n$$

are holomorphic.

We also say that F is a (holomorphic) embedding *if F is holomorphic, injective, and an immersion, that is, for every $0 \leq j \leq n$, denoting by*

$$\varphi_j \circ F = (F_1^j, \cdots, F_n^j)$$

with $F_1^j, \cdots, F_n^j : F^{-1}(U_j) \to \mathbb{C}$, we have that for every $x \in F^{-1}(U_j)$, there is some $0 \leq i \leq n$ so that

$$(dF_i^j)_x \neq 0.$$

In the rest of this section, we will show that every compact Riemann surface admits a holomorphic embedding into some complex projective space. The proof will be constructive, i.e. we will explicitly construct a holomorphic embedding. To do this, we present the following procedure of extending meromorphic functions into maps with codomain equal to a higher-dimensional complex projective space.

Example 3.3 (Constructing maps into complex projective space, cf. [For81, Paragraph 17.21])*.* Let X be a Riemann surface and let $f_0, \cdots, f_n \in \mathcal{M}(X)$ be non-zero meromorphic functions. We define the map

$$F := [f_0 : \cdots : f_n] : X \to \mathbb{CP}^n$$

as follows: for every $x \in X$, let (U, z) be a holomorphic chart centered at x, and let

$$k := \min_{0 \leq j \leq n} \operatorname{ord}_x(f_j).$$

Then, on U, we can write

$$f_j(z) = z^k g_j(z)$$

where each g_j is holomorphic near x and at least one g_j satisfies $g_j(x) \neq 0$. Then, for $z \in U$, we set

$$F(z) := [g_0(z) : \cdots : g_n(z)] \in \mathbb{CP}^n.$$

It is easy to check that this way, F is well-defined, independent of the coordinate chart and that it is holomorphic.

Now, the proof that every compact Riemann surface X (holomorphically) embeds into some complex projective space is simple to describe: we will consider a divisor D on X of large enough degree, take a basis $(f_0, \cdots, f_n)$ of $H^0(X, \mathcal{O}_D)$, and then the map

$$[f_0 : \cdots : f_n] : X \to \mathbb{CP}^n$$

will be the desired holomorphic embedding.

To prove the above, we first need to consider the following notion for the sheaf induced by a divisor.

Definition 3.4 (cf. [For81, Paragraph 17.18])**.** *Let X be a Riemann surface and let D be a divisor on X.*

We say that $\mathcal{O}_D$ is globally generated *if for every $x \in X$, there is some $f \in H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$ so that*

$$\mathcal{O}_{D,X} = \mathcal{O}_x \cdot f,$$

that is, for every germ $s \in \mathcal{O}_{D,x}$ in the stalk of $\mathcal{O}_D$ at x, there is some holomorphic map ψ defined on a neighbourhood U of x so that

$$s = (\psi \cdot f|_U)_x,$$

where $(\psi \cdot f|_U)_x$ is the germ of $\psi \cdot f|_U$ in $\mathcal{O}_{D,x}$.

For a Riemann surface X and a divisor D on X, notice that $\mathcal{O}_D$ is globally generated if and only if for every $x \in X$, there is some $f \in \mathcal{O}_D(X)$ with

$$\operatorname{ord}_x(f) = -D(x).$$

Proposition 3.5 (cf. [For81, Theorem 17.19])**.** *Let X be a compact Riemann surface of genus $g \geq 0$ and let D be a divisor on X with $\deg(D) \geq 2g$.*

Then, $\mathcal{O}_D$ is globally generated.

Proof. Let $x \in X$ and define the divisor

$$D' : X \to \mathbb{Z}, \quad D'(y) := \begin{cases} D(y) & \text{if } y \neq x \\ D(x) - 1 & \text{if } y = x \end{cases}$$

Observe that $\deg(D') = \deg(D) - 1$, and thus

$$\deg(D'), \deg(D) \geq 2g - 1.$$

Thus, by Theorem 2.1, we have

$$H^1(X, \mathcal{O}_D) = H^1(X, \mathcal{O}_{D'}) = 0.$$

Hence, the Riemann-Roch theorem implies that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_D) > \dim_{\mathbb{C}} H^0(X, \mathcal{O}_{D'}),$$

so there is some $f \in \mathcal{O}_D(X) \setminus \mathcal{O}_{D'}(X)$ By definition of the sheaves $\mathcal{O}_D$ and $\mathcal{O}_{D'}$, we necessarily have

$$\mathrm{ord}_x(f) = -D(x).$$

Hence, $\mathcal{O}_D$ is globally generated. □

With the above results, we can now prove that every compact Riemann surface (holomorphically) embeds into some complex projective space.

Theorem 3.6 (cf. [For81, Theorem 17.22])**.** *Let X be a compact Riemann surface of genus $g \geq 0$ and let D be a divisor on X with $\deg(D) \geq 2g + 1$. Let $f_0, \cdots, f_n$ be a basis for $H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$.*

Then, the map

$$[f_0 : \cdots : f_n] : X \to \mathbb{CP}^n$$

is a holomorphic embedding.

Proof. Denote by

$$F := [f_0 : \cdots : f_n].$$

We first show that F is injective. Let $x_1, x_2 \in X$ be so that $x_1 \neq x_2$. We need to show that $F(x_1) \neq F(x_2)$. Define the divisor

$$D' : X \to \mathbb{Z}, \quad D'(x) := \begin{cases} D(x) & \text{if } x \neq x_2 \\ D(x) - 1 & \text{if } x = x_2 \end{cases}$$

Observe that $\deg(D') = \deg(D) - 1 \geq 2g$, hence, by Proposition 3.5, the sheaves $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ are globally generated. Thus, there is some $f \in \mathcal{O}_{D'}(X)$ so that

$$\mathrm{ord}_{x_1}(f) = -D'(x_1) = -D(x_1). \tag{3.2}$$

By definition of D' and by assumption on f, we also have that

$$\mathrm{ord}_{x_2}(f) \geq -D'(x_2) = -D(x_2) + 1 \tag{3.3}$$

hence $f \in \mathcal{O}_D(X)$. Thus, we may write

$$f = \sum_{j=0}^{n} \lambda_j f_j \tag{3.4}$$

for some $\lambda_0, \cdots, \lambda_n \in \mathbb{C}$.

Let (U_1, z_1) and (U_2, z_2) be holomorphic coordinates centered x_1 and x_2, respectively. For $\ell \in \{1, 2\}$, put

$$k_\ell := \min_{0 \leq j \leq n} \mathrm{ord}_{x_\ell} f_j,$$

and observe that $k_\ell = -D(x_\ell)$ for $\ell \in \{1, 2\}$. Then, on U_ℓ, we may write

$$f_j(z_\ell) = z_\ell^{k_\ell} g_{\ell j}(z_\ell), \quad f(z_\ell) = z_\ell^{k_\ell} g_\ell(z_\ell),$$

for every $0 \leq j \leq n$ and every $\ell = \{1, 2\}$, where $g_{\ell j}$ and g_ℓ are holomorphic functions defined on U_ℓ. By (3.4), it follows that for every $\ell \in \{1, 2\}$, we have

$$\sum_{j=0}^{n} \lambda_j g_{\ell j}(x_\ell) = g_\ell(x_\ell).$$

Equations (3.2) and (3.3) imply that

$$g_1(x_1) \neq 0 \text{ and } g_2(x_2) = 0. \tag{3.5}$$

By definition of F (cf. Example 3.3), we have

$$F(x_1) = [g_{10}(x_1) : \cdots : g_{1n}(x_1)] \text{ and } F(x_2) = [g_{20}(x_2) : \cdots : g_{2n}(x_2)],$$

and thus $F(x_1) \neq F(x_2)$ because of (3.5). Hence, F is injective.

Secondly, we show that F is an immersion. Let $x_0 \in X$. Define the divisor

$$D' : X \to \mathbb{Z}, \quad D'(x) := \begin{cases} D(x) & \text{if } x \neq x_0 \\ D(x) - 1 & \text{if } x = x_0 \end{cases}$$

As in the proof of the first part of this theorem, the sheaf $\mathcal{O}_{D'}$ is globally generated by Proposition 3.5, hence there is some $f \in H^0(X, \mathcal{O}_{D'})$ with $\mathrm{ord}_{x_0}(f) = -D(X_0) + 1$. As before, we also have $f \in H^0(X, \mathcal{O}_D)$, and hence we may write

$$f = \sum_{j=0}^{n} \lambda_j f_j \tag{3.6}$$

for some $\lambda_0, \cdots, \lambda_n \in \mathbb{C}$. Then, we may choose holomorphic coordinates (U, z) centered at x_0 so that, on U,

$$f_j(z) = z^k g_j(z) \text{ and } f(z) = z^k g(z),$$

where

$$k := \min_{0 \leq j \leq n} \mathrm{ord}_{x_0}(f_j) = -D(x_0)$$

and $g_0, \cdots, g_n, g$ are holomorphic functions on U with $(g_0(x_0), \cdots, g_n(x_0)) \neq 0$ and g has a zero of order one at x_0. Let $0 \leq \ell \leq n$ be so that

$$g_\ell(x_0) \neq 0.$$

Then, with φ_ℓ defined as in (3.1), we compute, for $z \in U$ (after possibly shrinking U so that $g_\ell \neq 0$ on U),

$$(\varphi_0 \circ F)(z) = \left(\frac{g_0(z)}{g_\ell(z)}, \cdots, \frac{g_{\ell-1}(z)}{g_\ell(z)}, \frac{g_{\ell+1}(z)}{g_\ell(z)}, \cdots, \frac{g_n(z)}{g_\ell(z)} \right)$$

Denote by

$$F_j := \frac{g_j}{g_\ell}$$

for $j \in \{0, 1, \cdots, \ell-1, \ell+1, \cdots, n\}$. By (3.6), we have

$$\sum_{j \neq \ell} \lambda_j g_j = g - \lambda_\ell g_\ell,$$

hence

$$\sum_{j \neq \ell} \lambda_j F_j = \frac{g}{g_\ell} - \lambda_\ell.$$

Thus,

$$\begin{aligned} \sum_{j \neq \ell} \lambda_j (dF_j)_{x_0} &= \left(d \left(\frac{g}{g_\ell} - \lambda_\ell \right) \right)_{x_0} \\ &= \left(d \left(\frac{g}{g_\ell} \right) \right)_{x_0}. \end{aligned}$$

However, since $g_\ell(x_0) \neq 0$ and since g has a zero of order one at x_0, it follows from the chain rule that
$$\left(d\left(\frac{g}{g_\ell}\right)\right)_{x_0} \neq 0,$$
hence there is some $j \neq \ell$ so that
$$\lambda_j(dF_j)(x_0) \neq 0,$$
hence $(dF_j)(x_0) \neq 0$ and so F is an immersion at x_0. As x_0 was arbitrary, the proof is completed. □

Remark 3.7. It can be shown that every compact Riemann surface admits a holomorphic embedding into $\mathbb{CP}^3$.

Example 3.8 (cf. [For81, Exercise 17.6]). Let $\Gamma \subset \mathbb{C}$ be a lattice and consider the complex torus $\mathbb{C}/\Gamma$. The 1-form dz on $\mathbb{C}$ is invariant under translations. Hence, dz descends to a holomorphic 1-form on $\mathbb{C}/\Gamma$ that has no zeroes and no poles, i.e. has degree equal to 0. By Proposition 1.9, it follows that $\mathbb{C}/\Gamma$ has genus equal to 1.

Consider the divisor
$$D : \mathbb{C}/\Gamma \to \mathbb{Z}, \quad D(x) := \begin{cases} 3 & \text{if } x = 0 + \Gamma \\ 0 & \text{if } x \neq 0 + \Gamma \end{cases} \tag{3.7}$$
By definition,
$$\deg(D) = 3 > 1,$$
hence $H^1(\mathbb{C}/\Gamma, \mathcal{O}_D) = 0$ by Theorem 2.1. Thus, by the Riemann-Roch Theorem (cf. Theorem 1.4), we have
$$\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{O}_D) = 3. \tag{3.8}$$
Theorem 3.6 then implies that $\mathbb{C}/\Gamma$ embeds holomorphically into $\mathbb{CP}^2$. The dimension of this complex projective is also “minimal” in the sense that $\mathbb{C}/\Gamma$ does not (holomorphically) embed into $\mathbb{CP}^1$ (for purely topological reasons).

Let us construct an explicit embedding of $\mathbb{C}/\Gamma$ into $\mathbb{CP}^2$.

The *Weierstrass* $\wp$ *function of* Γ is defined as
$$\wp : \mathbb{C} \setminus \Gamma \to \mathbb{C}, \quad \wp_\Gamma(z) := \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right). \tag{3.9}$$
Observe that for $z \in \mathbb{C} \setminus \Gamma$ and $\omega \in \Gamma \setminus \{0\}$, we have
$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{2z\omega - \omega^2}{\omega^2(z-\omega)^2},$$
and the right-hand side is asymptotic to $|z|/|\omega|^3$ as $|\omega| \to \infty$. Hence, the series in (3.9) converges absolutely and uniformly in a neighbourhood of every point $z \in \mathbb{C} \setminus \Gamma$, and thus $\wp$ is a well-defined meromorphic function on $\mathbb{C}$. The definition of $\wp$ and the above arguments also imply that $\wp$ has poles of order 2 at every point of Γ.

The derivative of $\wp$ can be determined by differentiating the series (3.9) term-by-term, i.e.
$$\forall z \in \mathbb{C} \setminus \Gamma : \wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}, \tag{3.10}$$
and thus $\wp'$ is a meromorphic function on $\mathbb{C}$ with poles of order 3 at every point of Γ.

Notice also that, from (3.9) and (3.10), both $\wp$ and $\wp'$ are invariant under translation of the argument by an element of Γ, i.e. for every $z \in \mathbb{C} \setminus \Gamma$ and for all $\omega \in \Gamma$, we have
$$\wp(z+\omega) = \wp(z) \text{ and } \wp'(z+\omega) = \wp'(z).$$

Thus, both $\wp$ and $\wp'$ descend to meromorphic functions on $\mathbb{C}/\Gamma$ (by abuse of notation, we also denote these functions on $\mathbb{C}/\Gamma$ by $\wp$ and $\wp'$), and $0+\Gamma$ is their only pole, having order 2 and 3, respectively. In particular,

$$\wp, \wp' \in \mathcal{O}_D(\mathbb{C}/\Gamma) = H^0(\mathbb{C}/\Gamma, \mathcal{O}_D),$$

where D is the divisor defined in (3.7).

We now claim that the functions $\mathbf{1}, \wp$, and $\wp'$ form a basis for $H^0(\mathbb{C}/\Gamma, \mathcal{O}_D)$, where $\mathbf{1}$ is the constant function equal to 1. Let us prove this claim. It suffices to show that the functions $\mathbf{1}, \wp$, and $\wp'$ are $\mathbb{C}$-linearly independent as functions on $\mathbb{C}/\Gamma$. However, this is clear, as the poles of these three functions have a different order, and thus the claim is proven.

Hence, by the above claim and by (3.8), the set $\{\mathbf{1}, \wp, \wp'\}$ is a basis for $H^0(\mathbb{C}/\Gamma, \mathcal{O}_D)$, and thus, by Theorem 3.6, the map

$$[1 : \wp : \wp'] : \mathbb{C}/\Gamma \to \mathbb{CP}^2$$

is a holomorphic embedding.

We can say something more about this particular embedding of the complex torus $\mathbb{C}/\Gamma$. Define the quantities

$$g_2 := 60 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3 := 140 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}.$$

It is possible to show that $\wp$ and $\wp'$ (as meromorphic functions on $\mathbb{C}$) satisfy the algebraic relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Hence, the embedding $[1 : \wp : \wp']$ has image contained in the elliptic curve

$$\left\{[X : Y : Z] \in \mathbb{CP}^2 \;\middle|\; Z^2X = 4Y^3 - g_2YX^2 - g_3X^3\right\},$$

and one can also show that, in fact, the image of the map this map is the whole elliptic curve (as the map $[1 : \wp : \wp']$ is a non-constant holomorphic map between compact Riemann surfaces, it must be surjective, see [For81, Theorem 2.7]). Thus, the complex torus $\mathbb{C}/\Gamma$ is an elliptic curve.

References

[For81] O. Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer, 1981.