**1.1. The closure of the derivative operator.**
Define $\mathcal{C}_c^\infty((0,1)) = \{u \in \mathcal{C}^\infty((0,1)) \colon \operatorname{supp} u \subset (0,1) \text{ is compact}\}$, where the *support* of $u$ is the set $\operatorname{supp}(u) = \overline{\{t \in (0,1) \colon u(t) \neq 0\}}$.

**(a)** Let $A = \frac{\mathrm{d}}{\mathrm{d}t} \colon D(A) := \mathcal{C}_c^\infty((0,1)) \subset \mathcal{C}^0([0,1]) \to \mathcal{C}^0([0,1])$. What is the closure of $A$? That is, find $D(\bar{A})$.

**(b)** Let $A = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \colon D(A) := \mathcal{C}^\infty([0,1]) \subset \mathcal{C}^0([0,1]) \to \mathcal{C}^0([0,1])$. In this case, find again the closure $D(\bar{A})$.

**(c)** Let $X = Y = L^2([0,1])$, and define $A = \frac{\mathrm{d}}{\mathrm{d}t} \colon D(A) := \mathcal{C}^1([0,1]) \subset X \to Y$. Show that $A$ is closable.

**1.2. An operator that is *not* closable.**
Let $X = L^2(\mathbb{R})$, $Y = \mathbb{R}$. Let $f \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$. Set

$$A : D(A) = \{u \in L^2(\mathbb{R}) \colon \operatorname{supp} u \text{ is compact}\} \ni u \mapsto \langle u, f \rangle = \int_{-\infty}^{\infty} u(t)\overline{f(t)}\,\mathrm{d}t.$$

Show that $A$ is not closable.

**1.3. Closed sum.**
Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces and let

$$A : D(A) \subset X \to Y \text{ and } B : D(B) \subset X \to Y$$

be linear operators with $D(A) \subset D(B)$. Assume that there exist constants $0 \leq a < 1$ and $b \geq 0$ such that for all $x \in D(A)$ we have the inequality

$$||Bx||_Y \leq a||Ax||_Y + b||x||_X. \tag{1}$$

Show that if $A$ has closed graph then $(A+B) : D(A) \to Y$ has closed graph.

*Hint.* Given a sequence $(x_n)_{n\in\mathbb{N}} \in D(A)$, prove the estimate

$$(1-a)||A(x_n - x_m)|| \leq ||(A+B)(x_n - x_m)|| + b||x_n - x_m|| \tag{2}$$

**1.4. Closable inverse.**
Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces. Let $A : D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A} : D(\bar{A}) \to Y$ is injective. Show that the inverse operator $A^{-1} : \operatorname{ran}(A) \subset Y \to D(A) \subset X$ is closable and that its closure $\overline{A^{-1}}$ is the operator $\overline{A}^{-1} : \operatorname{ran}(\bar{A}) \to D(\bar{A})$.

*Hint*: Consider the image of the graph of $A$ under the map

$$\begin{aligned} \chi : X \times Y &\to Y \times X \\ (x, y) &\mapsto (y, x). \end{aligned}$$