10.1. Sufficient conditions for a solutions to an elliptic ODE.

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let $g^{ij} = g^{ji}, b^i, c \in \mathcal{C}^\infty(\bar{\Omega})$ for $1 \le i, j \le n$. Assume the usual ellipticity condition: there exists $\lambda > 0$ so that for all $x \in \Omega$ we have $\sum_{i,j=1}^n g^{ij}\xi_i\xi_j \ge \lambda|\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. Let us define

$$Lu(x) = -\sum_{i,j=1}^n \partial_i(g^{ij}(x)\partial_j u(x)) + \sum_{i=1}^n b^i(x)\partial_i u(x) + c(x)u(x). \tag{1}$$

Prove that the boundary value problem $Lu = f$, $u|_{\partial\Omega} = 0$, has a unique solution $u \in H_0^1(\Omega)$ for any given $f \in L^2(\Omega)$ under either of the following conditions:

1. $b^i = 0$, $c \ge 0$.
2. $c \ge 0$, and $\max_{i=1,\dots,n} \|b^i\|_{L^\infty(\Omega)} < \epsilon$ where $\epsilon > 0$ depends on Ω, g^{ij}, c.
3. $c = 0$, and for some function $0 < \gamma \in \mathcal{C}^\infty(\bar{\Omega})$ we have $b^i(x) = -\sum_{j=1}^n g^{ij}(x)\frac{\partial_j\gamma(x)}{\gamma(x)}$.
   Hint. Expand $-\frac{1}{\gamma}\partial_i(g^{ij}\gamma\partial_j u)$.
   Remark. This in particular covers the case of the Laplace–Beltrami operator if one takes $f = (\det(g^{ij}))^{-1/2}$.

10.2. Energy functional for non-linear Poisson equation with cubic term.

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain, and let $f \in \mathcal{C}^\infty(\bar{\Omega})$. The goal of this exercise is to show the existence of a unique solution $u \in \mathcal{C}^\infty(\bar{\Omega})$ of the equation

$$-\Delta u + u^3 = f, \quad u|_{\partial\Omega} = 0. \tag{2}$$

(a) For $u \in H_0^1(\Omega)$, define

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}u^4 - fu\,\mathrm{d}x. \tag{3}$$

Show that E is well-defined and continuous. *Hint.* Use Sobolev embedding to handle the u^4 term.

(b) Show that E is coercive (i.e. $E(u) \to \infty$ as $\|u\|_{H_0^1(\Omega)} \to \infty$ and w.s.l.s.c. (weakly sequentially lower semi-continuous, i.e. if $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then $E(u) \le \liminf_{k\to\infty} E(u_k)$. Deduce that E has a minimizer $u \in H_0^1(\Omega)$.

(c) Show that the minimizer of E is unique.

(d) Show that the minimizer $u \in H_0^1(\Omega)$ of E is a weak solution of the PDE (2).

(e) Show, using the PDE (2) and Sobolev embedding, that $u \in H^2(\Omega)$, and thus $u \in \mathcal{C}^0(\bar{\Omega})$. By repeatedly using the PDE, show inductively that $u \in H^k(\Omega)$, $k = 3, 4, \dots$, and conclude that $u \in \mathcal{C}^\infty(\bar{\Omega})$.

10.3. Rellich compactness for general domains.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. (We do not make any assumptions on the regularity of Ω.) Show that the inclusion map $H_0^1(\Omega) \to L^2(\Omega)$ is compact. Proceed as follows:

(a) Let $L > 0$ be such that $\Omega \subset (-L, L)^n$. Show that there is a well-defined, linear, continuous map $H_0^1(\Omega) \to H_0^1((-L, L)^n)$ given by extension by 0.

(b) The goal of this part is to show that the inclusion map $H^1(Q) \to L^2(Q)$ is compact. In order to do this, let $B \subset \mathbb{R}^n$ be a ball containing Q, and show that there exists a continuous extension operator $H^1(Q) \to H^1(B)$. Deduce the desired compactness statement from here.
Hint. One idea for the construction of the extension operator is to use the extension across $\mathbb{R}^{n-1} \times \{0\}$ from the lecture several times in order to extend a function in $H^1(Q)$ to an element of H^1 on the union of Q and its reflection along one of its boundary faces, and continue in this fashion to extend to larger and larger domains (given by unions of translates of Q).

(c) Let $u \in L^2(Q)$ and write

$$u(x) = \sum_{k \in \mathbb{Z}^n} u_k e^{ik \cdot x}$$

where $(u_k)_{k \in \mathbb{Z}^n}$ is square-summable. (You may take for granted that one can express u in this manner.) Show that $u \in H^1(Q)$ if and only if $\sum_{k \in \mathbb{Z}^n} (1 + |k|^2)|u_k|^2 < \infty$.

(d) Show that the inclusion map $H^1(Q) \to L^2(Q)$ is compact, and deduce that $H_0^1(Q) \to L^2(Q)$ is compact.

10.4. Min-max characterization of eigenvalues.

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and write $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ for the eigenvalues of $-\Delta$ (with Dirichlet boundary conditions), with multiplicity. Show that

$$\lambda_k = \inf_V \sup_{0 \neq u \in V} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2},$$

where the infimum is taken over all subspaces $V \subset H_0^1(\Omega)$ with $\dim V = k$. *Hint.* Do the case $k = 1$ first to get a feel for the problem.