11.1. A product of functions in $H_0^1(\Omega)$

Let $\Omega = (0, L_1) \times (0, L_2) \times \cdots \times (0, L_n)$ where $L_1, \ldots, L_n > 0$. Let $k_1, \ldots, k_n \in \mathbb{N}$. Show that the function

$$u(x_1, \ldots, x_n) = \prod_{j=1}^{n} \sin\Big(\frac{\pi k_j x_j}{L_j}\Big)$$

lies in $H_0^1(\Omega)$.

11.2. Decay rate of eigenfunction expansion of $-\Delta$ on $H_0^1(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^\infty$ domain, and let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ denote the eigenvalues of $-\Delta$ with Dirichlet boundary conditions, and denote by $\{u_k\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ the corresponding complete orthonormal basis of eigenfunctions. The goal of this exercise is to relate the norms of $H^s(\Omega)$ to the decay rate of the coefficients in expansions of functions on Ω into the basis $\{u_k\}$. Let $u \in L^2(\Omega)$, and write $u = \sum_{k=1}^{\infty} c_k u_k$ where $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

(a) Let $s = 2q$ where $q \in \mathbb{N}$. Show that $u \in H^{2q}(\Omega) \cap H_0^1(\Omega)$ if and only if $\sum_{k=1}^{\infty} |c_k|^2 \lambda_k^{2q} < \infty$.
(*Hint.*) Do this first for $q = 1$.

(b) Show that for $s = 2q$, there exists a constant $C = C(q)$ so that for all $u \in H^{2q}(\Omega) \cap H_0^1(\Omega)$ we have

$$C^{-1}\|u\|_{H^{2q}(\Omega)} \leq \sum_{k=1}^{\infty} |c_k|^2 \lambda_k^{2q} \leq C\|u\|_{H^{2q}(\Omega)}.$$

(c) Let $q \in \mathbb{N}$ be such that $2q > n/2$. Show the following pointwise bound for the k-th eigenfunction:

$$\|u_k\|_{L^\infty(\Omega)} \leq C\lambda_k^q.$$

where C depends only on Ω.

11.3. Asymptotics for the eigenvalues of $-\Delta$

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ denote the Dirichlet eigenvalues of $-\Delta$ on a smoothly bounded domain $\Omega \Subset \mathbb{R}^n$. Show the asymptotic formula

$$\lambda_k \sim 4\pi^2 \Big(\mathcal{L}^n(B_1(0))\mathcal{L}^n(\Omega)\Big)^{2/n} k^{2/n}$$

for the n-th eigenvalue. That is, show that the ratio of the left and the right hand side tends to 1 as $k \to \infty$.

11.4. Supremum bounds for eigenfunctions on compact sets. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $v \in H_0^1(\Omega) \cap C^\infty(\Omega)$ be an eigenfunction of the Laplace operator with $\lambda > 0$. The goal of this exercise is to prove that for any compact $K \subset \Omega$ we have that

$$||v||_{K,\Omega} := \sup_{x \in K} |v(x)| \leq C(K,\Omega)|\lambda|^{\frac{n}{4}+\frac{1}{2}}||v||_{L^2(\Omega)} \tag{1}$$

where $C(K,\Omega) > 0$ only depend on the sets K and Ω in $\mathbb{R}^n$.

(a) Let $\Omega'' \Subset \Omega' \Subset \Omega$. Let $\chi, \tilde{\chi} \in \mathcal{C}_c^\infty(\Omega)$ with $\chi \equiv 1$ on Ω'', $\text{supp}\chi \subset \Omega'$ and $\tilde{\chi} \equiv 1$ on $\text{supp}(\chi)$. Show that there exists a constant C, depending only on $\chi, \tilde{\chi}$, so that for $u \in H^2(\Omega)$ solving $-\Delta u = f \in H^k(\Omega)$, we have

$$\|\chi u\|_{H^{k+2}(\Omega)} \leq C\Big(\|\tilde{\chi} f\|_{H^k(\Omega)} + \|\tilde{\chi} u\|_{L^2(\Omega)}\Big).$$

(b) Let v be as above. Prove that for any $\chi \in C_c^\infty(\Omega)$ there exists a $C_2(k,\chi) > 0$ such that

$$||\chi v||_{H^k(\Omega)} \leq C_2(k,\chi)|\lambda|^{\frac{k}{2}}||v||_{L^2(\Omega)}. \tag{2}$$

(*Hint.*) Consider $|\lambda| > 1$.

(c) Prove equation (1).
(*Hint.*) You might find it useful to shortly state and prove the following Sobolev embedding for the compact set $K \subset \mathbb{R}^n$ and open $\Omega \subset \mathbb{R}^n$: for $k > \frac{n}{2}$, and $u \in C^0(\Omega) \cap H^k(\Omega)$ there exists a $C_3(k) > 0$ such that

$$\sup_{x \in K} |u(x)| \leq C_3(k)||u||_{H^k(\Omega)}.$$

11.5. The heat equation.

Let $\Omega \subset \mathbb{R}^d$ be a an open and bounded set with smooth boundary. Let $u_0 \in L^2(\Omega)$ be a given initial heat distribution. We would like to analyze the *heat equation*

$$\partial_t u(x,t) = \Delta_x u(x,t). \tag{3}$$

with boundary conditions

$$\begin{cases} u(x,t) = 0 \text{ for all } x \in \partial\Omega \text{ and } t > 0 \\ u(x,0) = u_0(x) \text{ for all } x \in \Omega. \end{cases} \tag{4}$$

Here u is a function of $x \in \Omega$ and $t \in \mathbb{R}_+$.

(a) Use the *principle of superposition* or otherwise to argue that one should attempt to solve the heat equation (3) with boundary values (4) using the Ansatz

$$u(x,t) = \sum_{n=1}^{\infty} a_n f_n(x) e^{\lambda_n t} \tag{5}$$

where here the $f_n \in C^\infty(\Omega) \cap H_0^1(\Omega)$ are the eigenfunctions of the Laplace operator on Ω with eigenvalues λ_n that form an orthonormal basis of $L^2(\Omega)$. Furthermore, the coefficients a_n are chosen such that $u_0(x) = \sum_{n=1}^\infty a_n f_n(x)$.

(b) In the following we want to make the Ansatz more precise. Let u be as in (5), show that

$$\lim_{t \searrow 0} ||u(\cdot,t) - u_0||_{L^2(\Omega)} = 0, \tag{6}$$

and that $u(\cdot,t) \in H_0^1(\Omega)$ for any $t > 0$. In this sense the boundary conditions (4) are satisfied a.e.

(*Hint.*) For the latter statement you can use Weyl's law and the fact (prove this!) that $H_0^1(\Omega) \ni g(x) = \sum_{n=1}^\infty a_n f_n(x)$ if and only

$$\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| < \infty. \tag{7}$$

(c) Let $K \subset \Omega$ be compact. Use exercise 3 and 4 to show that the series in (5) converges uniformily on K for fixed $t > 0$. Deduce that $u(\cdot,t)$ is conitnuous on Ω and that

$$\lim_{t \searrow 0} \sup_{x \in K} |u(x,t)| = 0. \tag{8}$$

(d) State an estimate for the derivatives of f_n that you expect to hold in analogy to exercise 4 equation (1). Use it to prove that $u \in C^\infty(\Omega \times \mathbb{R}_{>0})$.