12.1. The wave equation.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^\infty$ domain. Let $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ denote the Dirichlet eigenvalues of $-\Delta$, with multiplicity, and write $u_1, u_2, \ldots \in H_0^1(\Omega)$ for the corresponding eigenfunctions, normalized so that they form a complete orthonormal basis of $L^2(\Omega)$. In this problem, you may use that for $k = 1, 2$ there exists a constant C_k so that for $u \in H_0^1(\Omega)$ with $u = \sum_{j=1}^\infty c_j u_j$, we have

$$\|u\|_{H^k(\Omega)}^2 \le C_k \sum_{j=1}^\infty |c_j|^2 \lambda_j^{1/2}, \qquad \sum_{j=1}^\infty |c_j|^2 \lambda_j^{k/2} \le C_k \|u\|_{H^k(\Omega)}^2.$$

Let $\phi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\phi_1 \in H_0^1(\Omega)$. For $t \in \mathbb{R}$, define

$$\phi(t) := \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1$$

using the functional calculus of $-\Delta\colon H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$.

(a) Show that $\phi \in \mathcal{C}^0(\mathbb{R}; H^2(\Omega)) \cap \mathcal{C}^1(\mathbb{R}; H_0^1(\Omega)) \cap \mathcal{C}^2(\mathbb{R}; L^2(\Omega))$. (Here, we write $\mathcal{C}^0$ for continuous, but not necessarily uniformly bounded, functions, and similarly for $\mathcal{C}^1$.)

(b) Show that ϕ solves the initial value problem for the wave equation:

$$\begin{cases} (-\partial_t^2 + \Delta)\phi(t,x) = 0, \\ \phi(0,x) = \phi_0(x), \\ \partial_t \phi(0,x) = \phi_1(x). \end{cases}$$

12.2. Self-adjointness of $-\Delta$ with Dirichlet boundary conditions on general domains.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $D := \{u \in H_0^1(\Omega)\colon \exists f \in L^2(\Omega) \text{ s.t. } (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}\ \forall\, v \in H_0^1(\Omega)\}$. (That is, D consists of all H_0^1-functions whose Laplacian is, weakly, equal to an element of L^2.) The goal of this problem is to show that $-\Delta\colon D \subset L^2(\Omega) \to L^2(\Omega)$ is self-adjoint. Here $-\Delta : D \to L^2(\Omega)$ is defined as the operator that maps $u \in D$ to their corresponding $f \in L^2(\Omega)$ given in the description of D.

(a) Define the map $B\colon H_0^1(\Omega) \to H^{-1}(\Omega) = (H_0^1(\Omega))^*$ by $(Bu)(v) = (\nabla u, \nabla v)_{L^2(\Omega)} + i(u, v)_{L^2(\Omega)}$. Show that B is invertible. *Hint.* Show that B is a compact perturbation of the map $u \mapsto (v \mapsto (\nabla u, \nabla v)_{L^2(\Omega)})$, which we proved in class to be invertible. Then show that B is injective.

(b) Show that $-\Delta + i\colon D \to L^2(\Omega)$ is invertible. *Hint.* Show that its inverse is given by B^{-1}: this requires you to show that B^{-1} maps $L^2(\Omega)$ into D.

(c) Prove that $-\Delta\colon D \subset L^2(\Omega) \to L^2(\Omega)$ is self-adjoint.

(d) Show that $\sigma(-\Delta) = \sigma_p(-\Delta)$.

12.3. Spectrum of potentials on $\mathbb{R}^n$.

Let $n \geq 1$, and let $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

(a) Let $\lambda \in \mathbb{R}$, $\lambda < 0$. Show that $-\Delta - \lambda\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible.

(b) Recall from class that $-\Delta + V\colon H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint. Show that for $\lambda < 0$, the operator $-\Delta + V - \lambda\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm of index 0. Conclude that every $\lambda \in \sigma(-\Delta + V) \cap (-\infty, 0)$ is an eigenvalue (i.e. lies in the point spectrum).

(c) Let $A\colon D(A) \subset H \to H$ be a self-adjoint operator on a Hilbert space H. Show that $\sigma(A) \cap (-\infty, 0) \neq \emptyset$ if and only if there exists $u \in D(A)$ so that $(Au, u) < 0$.

(d) Assume that $V \leq 0$ and $V \not\equiv 0$. Show, using the criterion in part (3), that there exists a constant $C_0 > 0$ so that for all $C > 0$, we have $\sigma(-\Delta + CV) \cap (-\infty, 0) \neq \emptyset$.
Remark. One can show that the operator $-\Delta + CV$ has continuous spectrum $[0, \infty)$; and by part (2), its spectrum in $(-\infty, 0)$ consists entirely of point spectrum.