2.1. An equivalence for closed surjective operators.

Let X, Y be Banach spaces, let $A : D_A \subset X \to Y$ be a linear densely defined closed operator, with dual operator $A^* : D_{A^*} \subset Y^* \to X^*$. Prove that the following are equivalent:

(a) A is surjective.

(b) A^* is injective, and $\mathrm{Im}(A^*)$ is closed.

(c) There exists $c_0 > 0$ such that for all $y^* \in D_{A^*}$ we have

$$c_0 ||y^*||_{Y^*} \leq ||A^* y^*||_{X^*}. \tag{1}$$

2.2. Self adjoint extensions of $i\frac{\mathrm{d}}{\mathrm{d}t}$.

Fix $\alpha \in \mathbb{R}$. Let $A_\alpha = i\frac{\mathrm{d}}{\mathrm{d}t}$ with domain $D(A_\alpha) := \{u \in H^1(0,1) \colon u(1) = e^{i\alpha}u(0)\} \subset L(0,2)$.

(a) Show that A_α is self-adjoint for all $\alpha \in \mathbb{R}$.

For the rest of the exercise we aim to prove that all self-adjoint extensions of $i\frac{\mathrm{d}}{\mathrm{d}t}$ are equal to A_α for some $\alpha \in \mathbb{R}$.

(b) Let $H_0^1(0,1) := \{u \in H^1(0,1) \colon u(0) = u(1) = 0\}$. Show that the map $H^1(0,1) \ni u \mapsto (u(0), u(1)) \in \mathbb{C}^2$ is surjective with kernel $H_0^1(0,1)$, and deduce that
$\dim H^1(0,1)/H_0^1(0,1) = 2$.

(c) Let $B_0 = i\frac{\mathrm{d}}{\mathrm{d}t}$, but with domain $D(B_0) = H_0^1(0,1)$. Show that if B is a self-adjoint extension of B_0, then $\dim D(B_0^*)/D(B) = 1$.
(*Hint.* Recall that the adjoint of $B_0 = i\frac{\mathrm{d}}{\mathrm{d}t} \colon D(B_0) \subset L^2(0,1) \to L^2(0,1)$ is given by $B_0^* = i\frac{\mathrm{d}}{\mathrm{d}t}$ with $D(B_0^*) = H^1(0,1)$. Thus, part (b) gives $\dim D(B_0^*)/D(B_0) = 2$.) Conclude that there exists a nonzero linear functional $\lambda \colon \mathbb{C}^2 \to \mathbb{C}$ so that $D(B) = \{u \in H^1(0,1) \colon \lambda(u(0), u(1)) = 0\}$.)

(d) If $B \supset B_0$ is self-adjoint, show that $B = A_\alpha$ for some $\alpha \in \mathbb{R}$.
(*Hint.* Write $\lambda(u_0, u_1) = a_0 u_0 + a_1 u_1$ where $a_0, a_1 \in \mathbb{C}$ are not both zero. Write out the condition for self-adjointness and compare this with what you get via integration by parts.)

2.3. Spectrum and adjoint of an operator on $l^2(\mathbb{Z})$. Let $(q_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence of complex numbers. Set $D = \{a = (a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \colon (q_n a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\}$. Define $A \colon D \subset \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $A(a_n)_{n\in\mathbb{Z}} = (q_n a_n)_{n\in\mathbb{Z}}$. Prove the following.

(a) Show that A is closed.

(b) Compute the adjoint of A. (This includes the determination of its domain.) Conclude that A is self-adjoint when $q_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$.

(c) Compute the spectrum of A. Give an example of a sequence $(q_n)_{n\in\mathbb{Z}}$ so that $\sigma_{\mathrm{c}}(A) \neq \emptyset$.

2.4. The adjoint operator is *always* closed.

Let H be a Hilbert space, and define the operator $V\colon H \times H \to H \times H$ by $V(x, y) = (-y, x)$. Let $A\colon D(A) \subset H \to H$ be a densely defined operator. Show that $\Gamma_{A^*} = (V(\Gamma_A))^{\perp}$.

Remark. Since orthogonal complements are always closed, this gives another proof that A^* is always closed.