3.1. Spectrum of a self-adjoint operator on $\mathbb{S}^1$

Let $V \in L^\infty(\mathbb{S}^1)$ be real-valued. We write $D_\theta = -i\frac{\mathrm{d}}{\mathrm{d}\theta}$.

(a) Prove that $P = D_\theta^2 + V(\theta)$ is self-adjoint with domain $H^2(\mathbb{S}^1)$. (Hint: Show this first for $V = 0$ by proving that the operators $D_\theta^2 \pm i\colon H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are invertible. For general V, prove the invertibility of $D_\theta^2 + V \pm i$ using Fredholm theory.)

(b) Show that $\sigma(P) = \sigma_p(P)$ is a discrete subset of $\mathbb{R}$ which accumulates only at infinity, and that there exists a complete orthonormal basis of $L^2(\mathbb{S}^1)$ consisting of eigenfunctions of P.

3.2. Spectral calculus for commuting self-adjoint operators

Let H be a separable Hilbert space, and let $A, B \in L(H)$ be to self-adjoint operators which commute (that is, $AB = BA$). Let $f, g \in \mathcal{B}^\infty(\mathbb{R})$ be two bounded Borel measurable functions. Show that $f(A)$ and $g(B)$ commute, that is, $f(A)g(B) = g(B)f(A)$. Conclude that any two spectral projectors of A and B commute.

3.3. Resolvents to characterize the spectral measure

Let H be a separable Hilbert space, and let $A \in L(H)$ be self-adjoint. Denote by $R(z) = (z - A)^{-1}$ its resolvent. Let $a < b$, and let $f \in \mathcal{C}_c^0((a, b))$. (That is, f is continuous and has compact support in (a, b).) Let $u \in H$. Show that

$$\lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \Big(\big(R(\lambda - i\epsilon) - R(\lambda + i\epsilon)\big)u, u\Big) f(\lambda)\,\mathrm{d}\lambda = \big(f(A)u, u\big).$$

Conclude that $\frac{1}{2\pi i}((R(\cdot - i\epsilon) - R(\cdot + i\epsilon))u, u)$ converges to the spectral measure of u as $\epsilon \to \infty$ in the weak* topology.

3.4. Diagonalization of $i\frac{\mathrm{d}}{\mathrm{d}t}$

Find a finite measure space (M, μ), a measurable function $f\colon M \to \mathbb{R}$, and a unitary map $U : L^2([0, 1]) \to L^2(M, \mathrm{d}\mu)$ which diagonalizes the self-adjoint operator $A = i\frac{\mathrm{d}}{\mathrm{d}t}$ with domain $D(A) = H_0^1([0, 1]) = \{u \in H^1([0, 1])\colon u(0) = u(1)\}$, that is

$$UAU^{-1} = T_g, \tag{1}$$

where $T_g : (L^2(M, \mathrm{d}\mu) \to L^2(M, \mathrm{d}\mu)$ is multiplication by g. (This is an explicit special instance of the general spectral theorem for unbounded self-adjoint operators, theorem $T.11$ to be proved in class shortly.)