4.1. Explicit form of a unitary transformation and a self-adjoint operator

(a) Let $A = i\frac{\mathrm{d}}{\mathrm{d}t}$ with domain $D(A) = H^1_0([0,1]) = \{u \in H^1([0,1]) \colon u(0) = u(1)\}$. Recall that A is self-adjoint. Let $u \in L^2([0,1])$. Write down an explicit formula for $(e^{itA}u)(x)$.

(b) Do the same for the self-adjoint operator $A_\alpha = i\frac{\mathrm{d}}{\mathrm{d}t}$ with domain $D(A_\alpha) = X_\alpha := \{u \in H^1([0,1]) \colon u(0) = e^{i\alpha}u(1)\}$. *Hint.* Given your intuition from the first part, try to guess the result, and then prove that it is correct.

4.2. Cauchy's formula for the spectrum of self-adjoint operators

Let H be a complex separable Hilbert space. Let $A \in L(H)$ be self-adjoint. Let $U \subset \mathbb{C}$ be an open neighborhood of $\sigma(A)$, and suppose $f \colon U \to \mathbb{C}$ is holomorphic. Let $\gamma \subset U$ be a piecewise smooth curve winding once around $\sigma(A)$ counterclockwise. (For example, if $\sigma(A) \subset [a,b]$, then for small $\epsilon > 0$ we can take γ to be the concatenation of $[a-\epsilon, b+\epsilon] - i\epsilon$, $(b+\epsilon) + i[-\epsilon, \epsilon]$, $[a-\epsilon, b+\epsilon] + i\epsilon$, and $(a-\epsilon) + i[-\epsilon, \epsilon]$.) Show that

$$f(A) = \frac{1}{2\pi i} \oint_\gamma f(z)(z-A)^{-1}\,\mathrm{d}z.$$

4.3. Analicity of the resolvent mapping

Let H be a complex separable Hilbert space. Let $A \colon D(A) \subset H \to H$ be densely defined and closed. Write $R_z = (z-A)^{-1}$ for the resolvent of A when $z \in \rho(A)$

(a) Show that for $z, w \in \rho(A)$, we have

$$R_z - R_w = (w-z)R_z R_w.$$

(b) Show that the resolvent set $\rho(A)$ is open.

(c) Show that $\rho(A) \ni z \mapsto R_z(A) = (z-A)^{-1} \in L(H)$ is an analytic operator-valued function.

4.4. Heat equation and the exponential map

Let H be a complex separable Hilbert space. Let $A \colon D(A) \subset H \to H$ be self-adjoint and bounded from below (spectrum), i.e. $\sigma(A) \subset [C, \infty)$ for some $C > -\infty$. Let $u_0 \in D(A)$ and define $u(t) := e^{-tA}u_0$ for $t \geq 0$. Prove the following statements:

(a) $\|u\|_H \leq e^{-tC}\|u_0\|_H$ for $t \geq 0$.

(b) $u \in \mathcal{C}^1([0,\infty); H) \cap \mathcal{C}^0([0,\infty); D(A))$, where we equip $D(A)$ with the graph norm $\|u\|_{D(A)}^2 := \|u\|_H^2 + \|Au\|_H^2$ (so it is a Hilbert space).

(c) u satisfies the *heat equation*

$$\begin{cases} \frac{\partial u}{\partial t}(t) = -Au(t), \quad t \geq 0, \\ u(t) = u_0. \end{cases}$$