5.1. The p-energy functional.

Let $\emptyset \neq \subset \mathbb{R}^n$ be open, bounded and regular, $2 \leq p < \infty$ and $g \in C^2(\partial\Omega)$. Consider $E_p : W^{1,p}(\overline{\Omega}) \to \mathbb{R}$ as

$$E_p(u) := \int_\Omega |\nabla u|^p dx, \quad \text{and} \quad \mathfrak{U} := \{u \in C^2(\bar{\Omega}) \mid u_{\partial\Omega} = g\}$$

(a) Determine whether there is at most one function $u \in \mathfrak{U}$ satisfying

$$E_p(u) = \inf_{v \in \mathfrak{U}} E_p(v).$$

(b) Derive the partial equation satisfied by the minimisers of u of E_p as in **(a)**.

(c) Prove that for every $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega}$ the inequality

$$\int_\Omega |\nabla u|^p dx \leq C_{p,n} \left(\int_\Omega |u|^p dx \right)^{1/2} \left(\int_\Omega |D^2 u|^p dx \right)^{1/2}$$

5.2. Weak derivate in $L^p(\Omega)$.

(a) Let $\Omega \subset \mathbb{R}^n$ be open, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ be a multi-index and $|\alpha| = \sum_{k=1}^n \alpha_k$. Let $u \in L^1_{loc}(\Omega)$. Given $1 < p \leq \infty$, let $1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $D^\alpha u$ exists as a weak derivative in $L^p(\Omega)$ if and only if there exists a $C > 0$ such that

$$\left| \int_\Omega u D^\alpha \phi dx \right| \leq C ||\phi||_{L^q(\Omega)} \quad \text{for all} \quad \phi \in C_c^\infty(\Omega). \tag{1}$$

(b) The assumption $p \neq 1$ in **(a)** is necessary: prove that $u = \chi_{]0,1[} \in L^1(\mathbb{R})$ satisfies

$$\left| \int_\Omega u \phi' dx \right| \leq C ||\phi||_{L^\infty(\mathbb{R})}, \tag{2}$$

with some $C > 0$ but $u \notin W^{1,1}(\mathbb{R})$, i.e. u does not have a weak derivative in $L^1(\mathbb{R})$.

5.3. Weak derivative of a conic function.

Let $\Omega := B_1(0) \subset \mathbb{R}^2$ and consider the function $u(x, y) = 1 - \sqrt{x^2 + y^2}$ (whose graph is a reversed ice-cream cone).

(a) Determine the values $p \in [1, \infty]$ for which exist, in $L^p(\Omega)$, the weak partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

(b) For such values of p compute, as a function p, the value of $||\nabla u||_{L^p(\Omega)}$.

5.4. A closedness property.

Let $I :=]a, b[$ for $-\infty \leq a < b \leq \infty$. Let $u \in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(I)$ with $||u - u_k||_{L^p(I)} \to 0$ as $k \to \infty$.

(a) If $1 < p \leq \infty$, prove that $u \in W^{1,p}(I)$.

(b) Is the assumption $p \neq 1$ in part **(a)** necessary?