6.1. The Dirichlet problem on an interval.

Let $I = (a, b)$ be a bounded interval. Given $f \in C^0(\bar{I})$ we look for a $u \in C^2(\bar{I})$ with

$$\begin{cases} -u'' = f \quad \text{on} \quad I \\ u(a) = 0 = u(b) \end{cases} . \tag{1}$$

The problem above is called a *Dirichlet* problem if we are considering vanishing boundary conditions. A *weak* formulation of the problem above is then stated as finding a solution $u \in H_0^1(I)$ to the equation

$$\int_I u'v'dx = \int_I fvdx \quad \text{for all} \quad v \in H_0^1(I), \tag{2}$$

where $u' \in L^2(I)$ and $v' \in L^2(I)$ are the weak derivates of u and v respectively. Prove the following:

(a) Give a (short) argument as to why there exists a weak solution to the Dirchlet problem, i.e. a $u \in H_0^1(I)$ that solves (2).

(b) Show that if $u \in H_0^1(I)$ solves (2) it solves the Dirichlet problem in the strong sense, i.e. u lies in $C^2(\bar{I})$ and solves (1), including boundary conditions.

6.2. The Neumann problem on an interval.

Let $I = (a, b)$ be a bounded interval. Given $f \in C^0(\bar{I})$ we look for a $u \in C^2(\bar{I})$ with

$$\begin{cases} -u" = f \quad \text{on} \quad I \\ u'(a) = 0 = u'(b) \end{cases} . \tag{3}$$

A problem with vanishing boundary conditions on u' is called a *Neumann* problem. Direct integration of (3) over I immediately gives us the the equality

$$\int_I f(x) = 0. \tag{4}$$

At the same time we know that any solution if $u \in C^2(\bar{I})$ that solves (3) then $u + c$ with $c \in \mathbb{R}$ solves (3) as well. To incorporate the Neumann boundary conditions, we consider the space $X \subset H^1(I)$ as

$$X = \{u \in H^1(I) : \bar{u} = 0\}^1, \tag{5}$$

where $\bar{u}$ is the average of u given by

$$\bar{u} = \frac{1}{|I|} \int_I u(x)dx, \tag{6}$$

[1]Notice that X is a closed subspace of $H^1(I)$

in order to find a solution $u \in X$ to the weak formulation of (3) given as

$$\int_I u'v'dx = \int_I fvdx \quad \text{for all } v \in H^1(I). \tag{7}$$

To this end let us prove the following:

(a) X is a Hilbert space with scalar product

$$(u,v)_X \int_I u'v'dx, \quad \text{for all} \quad u,v \in X. \tag{8}$$

Hint. Show that if $u \in X$, u satisfies a "Poincaré"-like inequality; i.e. there exists a $C > 0$ such that

$$||u||_{L^2} \leq C \cdot ||u'||_{L^2}, \tag{9}$$

where C will depend only on I.

(b) Now prove that if $f \in C^0(\bar{I})$ satisfies (4) then there exists a unique weak solution $u \in X$ such that u solves (7).

(c) Show that for a $u \in X$ that solves (7, we have in fact that $u \in C^2(\bar{I})$ and solves (3) in the strong sense.

Remark. We note two important features in the above exercises. First of all the above implies that on the interval, a weak solution to the Dirichlet or Neumann problem is *always* a strong solution as well, i.e. $u \in C^2(\bar{I})^2$. Furthermore, we have now seen two good examples of the fact that for a 2nd order "PDE" it is enough to consider a solution in Sobolev spaces of order H^1 for the weak formulation of the problem, as discussed in the lecture.

6.3. Equivalent characterizations of $W^{1,p}(\Omega)$ Let $1 < p \leq \infty$. In this exercise, prove the generalization of theorem $T.19$ to an open bounded $\Omega \subset \mathbb{R}^n$, i.e. prove the following are equivalent:

(a) $u \in W^{1,p}(\Omega)$;

(b) There exists a $C > 0$ such that for all $\phi \in C_c^\infty(\Omega)$ we have

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} dx \right|, 1 \leq i \leq n, \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \tag{10}$$

[2] In fact, it will turn out that u will be smooth...

(c) There exists a $C > 0$ such that for all $\Omega' \Subset \Omega$, such that for all $h \in \mathbb{R}^n$, $|h| < d(\Omega, \Omega')$ we have

$$||\tau_h u - u||_{L^p(\Omega')} \leq C|h|, \tag{11}$$

where $\tau_h u(x) := u(x+h)$, for all $x \in \Omega'$.

For the above cases, indicate also explicitly what the constant $C > 0$ is.

6.4. Chain rule for Sobolev functions

Let $G \in C^1(\mathbb{R})$ with $G(0) = 0$ and $|G'(s)| \leq L$, and let $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$. Prove that $G \circ u \in W^{1,p}(\Omega)$ with

$$\nabla(G \circ u) = (G' \circ u) \cdot \nabla u \in L^p(\Omega) \tag{12}$$